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COSMOLOGICAL MODELS WITH WEAKLY TURBULENT FLUCTUATIONS OF THE METRIC SUPERIMPOSED ON A HOMOGENEOUS AND ISOTROPIC BACKGROUND METRIC

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THESIS

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by

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ABSTRACT

Uniform and isotropic mathematical models of the expanding universe usually predict an initial singularity of infinite mass density and space curvature. To study possible mechanisms which would avoid the occurance of these singularities, non-uniform cosmological models based on Einstein's field equations are investigated in which random perturbations of long wave lengths are superimposed on the Robinson-Walker metric of the unperturbed models. Techniques of fluid turbulence theory, used to describe random fields by a hierarchy of central moments of the random perturbations, are applied to describe the dynamics of these moments. For the case of small perturbations the hierarchy is truncated and solutions are found. The solutions are either growing or decaying perturbations leading to R^m extra terms in the usual cosmological equations for the curvature radius R. The result agrees with the small perturbation Fourier series expansion analysis which exists in the literature. Based on the upper limit of the anisotropy of the 3° K background radiation, the growing perturbation model predicts a maximum expansion even for k=0, Euclidean spaces. The decaying perturbation solutions give extra terms of the form $1/R^m$ with $m>4$ in the cosmological equations and indicate that the mechanism of long wave random perturbations may prevent the original singularity and make oscillatory models possible.

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I. INTRODUCTION

Perhaps one of the most exciting discoveries of the Twentieth Century thus far is Hubble's now famous discovery (1936) that the relative shift in wavelength $\frac{\Delta\lambda}{\lambda}$ in the spectral emissions of each star is proportional to its astronomical distance L [1]. Modern cosmology generally recognizes this phenomenon to be the consequence of an "expanding universe" whose dynamics are determined by the Einstein field equations [2, 21]. The concept of an expanding universe presents various problems to those who are involved with the systematic study of our universe as an entity. Because the term "expanding universe" implies that all distances between all points are on the average increasing, it is natural to ask if our universe was ever in a condition in which all lengths were zero. It is also natural to ask if this so-called expansion is homogeneous and isotropic at all points in the universe.

If the answer to the first question is yes, then it is apparent that sometime in the distant past the mass density of the universe was infinite. But Wheeler [20] has pointed out that such a state is rather dubious. He calls attention to the fact that it is possible to construct a "length" L^* by a combination of the gravitational constant G, the light velocity c, and the Planck constant $\hbar$ such that

$$L^* = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} = 1.6 \times 10^{-33} \text{ cm.} \tag{1-1}$$

He suggests that this length might be a characteristic length that is associated with a quantum theory of gravitation. He points out that because of the smallness of this length, quantum gravitational effects, though negligible now, could in principle be important during the very earliest stages of the expansion of our universe [5]. Hence, it is believed that the undesirable singularity associated with the infinite density might be avoided. One reason, he says, why such quantum effects might be significant during the highly contracted phase of our universe is that the estimated cross section for pairs of gravitons to produce pairs of electrons and pairs of photons is comparable when the wavelength of the compacted gravitational radiation, which is assumed to be present, is of the order of L^*. In order to describe such a situation, it is clear that a quantum theory must be used to treat this obviously nonclassical phenomenon [20].

Various workers have begun the task of incorporating quantum effects into gravitational theory. Some investigators (Misner [16] and Parker [17] for example) do this by various canonical procedures, but others have taken a different approach. Woehler [34] suggests that it might be fruitful to postpone a correct quantum theory of gravitation and try a semiclassical approach. A way he suggests to do this is to presume that the field variables $g_{\mu\nu}$ (called the

metric tensor) of Einstein's field equations randomly fluctuate with position and are characterized by a "mixing length" L, which for quantum fluctuations would take the value L*. By considering the average of the field equations, he suggests that it might be possible to avoid singular solutions if the extra source term, which appears as a consequence of the averaging and the nonlinearity of the field equations, is dominant during the early phase of high contraction of our universe. Ginzburg and other co-workers [7] use a similar approach. They attempt to estimate the functional form of this extra source term by expanding this source term in a series expansion of the average curvature tensor. It is interesting to note that both the techniques of Woehler and Ginzburg are reminiscent of an earlier technique used by Welton [33] to estimate the Lamb Shift by a semiclassical model that also involves fluctuations, but of the position variable of the electron.

However, the point in the above discussion that is germane to the introduction of this thesis is that there are today serious considerations of using a randomly fluctuating metric tensor $g_{\mu\nu}$ as a prelude to developing a realistic quantum theory of gravitation.

There is also another aspect of cosmology that requires the use of a nonuniform fluctuating metric tensor $g_{\mu\nu}$. This consideration is brought out by the second question that asks if the expansion is everywhere homogeneous and isotropic. It is a matter of history that one of the first

attempts to describe the universe from a cosmological point of view assumes that all matter and all space are homogeneous and isotropic on a sufficiently large scale. This theoretical assertion is called the Cosmological Principle and was formulated in 1935 [2, 21, 22].

Although the use of the Cosmological Principle appears regularly in the literature, its criticism is still very much alive today. De Vaucouleurs [31], in a recent paper, points out that because current observations have as yet failed to demonstrate the existence of that sufficiently large scale beyond which space is homogeneous and isotropic, it seems unlikely that it would be easily found (if such a length scale exists at all) by a slightly deeper astronomical survey.

Because of this and other arguments, many workers have been led to consider nonuniform cosmological models. One way this is done is to consider the dynamics of an anisotropic universe which has two equivalent and one inequivalent directions in space. The dynamics of such models can be determined exactly by solving the appropriate field equations [27]. Another method to construct nonuniform cosmological models is to consider small linear perturbations away from a spatially homogeneous and isotropic cosmological metric. Sachs and Wolfe [24] do this and are able to express these perturbations as a function of time. Their solutions contain some arbitrary functions of the three spatial coordinates which are determined by initial conditions of a given

arbitrary perturbation field. These functions then are generating functions which correspond to a given nonuniform model.

Again the important point for the reader of this paper to note is that cosmological models with a variety of nonuniformities are being actively considered by modern cosmologists. These nonuniformities may be zero point quantum mechanical fluctuations or they may be fluctuations due to a large scale uneven mass distribution. In fact, Ginzburg and others [7] point out that both types of fluctuations are described by the same formalism, a metric tensor that fluctuates randomly about its average value. Matzner [14] points out that this formalism is appropriate for a universe whose average background is uniform but contains gravitational radiation of an arbitrary size L. Clearly, some techniques are needed which describe these random perturbations of arbitrary magnitude and size L in some natural way.

II. MOTIVATING PHILOSOPHY OF THE MODEL

In 1934, Milne and McCrea chose to describe our universe in terms of a Newtonian model. This model treats the universe as a streaming, uniform fluid. Surprisingly, they showed that such a model is in many respects equivalent to the models of relativistic cosmology [2]. Their model has the distinct advantage of being mathematically much less complex than models developed within the framework of general relativity. Another advantage is that fluid flows have been well studied and it would seem, therefore, that some of the fruits of fluid mechanics might be transferable to the field of cosmology.

Such a possibility has, in part, been the motivation for preparing this dissertation. In particular, the study of turbulent fluid flows which deals with random perturbations of fluid flow fields would seem, on the basis of the above discussion, to be relevant to the study of random perturbations of the "flow field" associated with the universe.

It is well known that fluid flows obey the Navier-Stokes equation. This equation relates the velocity field $\vec{V}$ to a pressure field P associated with the flow. It is also well known that if both the velocity and pressure fields are decomposed into an average field plus a fluctuation field such that

$$\vec{V} = \langle\vec{V}\rangle + \vec{v}$$

$$P = \langle P\rangle + p \tag{2-1}$$

where < > denotes "average," then the Navier-Stokes equation can be placed into a form which is symbolically identical with the Navier-Stokes equation for $\vec{V}$ and P but with $\vec{V}$ replaced by $\langle\vec{V}\rangle$ and P replaced by $\langle P\rangle$. Owing to the non-linearity of the Navier-Stokes equation, additional source terms which involve the correlations $\langle v_i v_j\rangle$ appear (where i and j denote the i and j^{th} components of the fluctuation $\vec{v}$) [4]. These extra terms are called the Reynold Stresses and may be interpreted as additional stresses which act upon $\langle\vec{V}\rangle$.

There are a variety of methods that fluid dynamicists use to determine the functional form of the Reynold Stresses. An important way is to generate a hierarchy of moment equations in $\vec{v}$ by multiplying the unaveraged Navier-Stokes equation with all orders of $\vec{V}$ and averaging. Because of the nonlinearity, each new equation so generated contains a moment of one order higher than what would be present if the Navier-Stokes equation were linear. Although any number of moment equations could in principle be generated, there always exists one more unknown than there are equations. This problem is called the closure problem of turbulence theory. Much research has gone into finding ways to replace this indeterminate, infinite set of equations by a plausible determinate, finite set so that useful information about turbulant flow patterns might be obtained [3]. However, in

spite of the difficulties associated with the closure problem, it should be pointed out that the turbulence approach, which is a sufficiently general formalism, can in principle describe any sort of random perturbation field of any magnitude. (A short summary of turbulence theory of fluids is given in Appendix A.)

The Einstein Field equations are likewise nonlinear differential equations. Their field variables are the metric tensor $g_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$. Because certain aspects of the field equations of general relativity resemble those of fluid mechanics, it is not unreasonable to presume that the Einstein field equations could be expressed in terms of the average metric tensor $\langle g_{\mu\nu}\rangle$, the average energy-momentum tensor $\langle T_{\mu\nu}\rangle$, and some residual terms that appear in a way analogous to the Reynold Stresses owing to nonlinearities. It might also be possible to generate a moment hierarchy which can be used for the determination of these extra terms if an appropriate closure technique could be found.

It is this speculation therefore, based on the similarities between fluid mechanics and general relativity, that serves as the motivation for constructing such a technique which might be useful in solving perturbation problems of general relativity. The remainder of this dissertation is a reply to this conjecture.

III. INITIAL DESCRIPTION OF THE MODEL

Given that it is possible to generate a hierarchy of meaningful moment equations from the Einstein field equations, it is necessary to truncate them in some manner. Then a solution for the "Reynold Stresses" can be determined which will be found to contain second order moments in the perturbed metric and its derivatives. Just as in fluid turbulence theory, the closure of the moment hierarchy is nontrivial because it requires special restrictive assumptions concerning the nature of the random perturbation field and does not just "fall out" of the theoretical fra framework.

In order to see how this truncation might be accomplished, consider first the following three important truncation procedures which are used in fluid turbulence theory to close its unlimited set of equations which contain moments of all orders of the velocity $\vec{v}$ [3].

1. Phenomenological approximations - These involve postulating (on dimensional grounds or by intuition) an extra relation between various moments to bring the number of equations up to the number of unknowns.
2. Higher-order-moment-discard approximation - This technique simply discards the higher order moments because of smallness or some other reason. This reduces the number of unknowns to the number of available equations.

3. Cumulant-discard approximation - This procedure relates a higher order moment to products of lower order moments and thereby effectively establishes an extra equation.

All these closure methods are used in fluid turbulence theory and so all might conceivably be acceptable candidates for closing a moment hierarchy in general relativity as well.

If it is desired to describe cosmological models with a "small" degree of anisotropy and inhomogeneity, then it would seem that the appropriate truncation would be number two, the higher-order-moment-discard approximation. This choice is reasonable because it would be expected that a moment of order n+1 would be negligible compared to a moment of order n. Such weakly perturbed models are considered, for example, by Sachs and Wolfe [24] as previously stated.

However, not all cosmological models are adequately described by small perturbations. In a recent paper by Woehler [34], a model is presented which assumes quantum fluctuations of the metric tensor to be as large as the average metric tensor itself. Certainly the higher-order-moment-discard approximation is not adequate to apply to this model because all moments of all orders are important, especially near the singularity. For this type of model, the first or third truncation alternative might be useful.

Therefore, an attempt to construct a general perturbation technique based on a moment hierarchy scheme which is able to treat fluctuations of all shapes and sizes is discouraged by the fact that the method of truncation depends

upon the type of fluctuation that is involved. In fact, even the solutions of the moment hierarchy could, in general, be quite different owing to the fact that different moment equations could result from different methods of truncation.

So it seems that a reasonable way to proceed is to consider a model with a certain "kind" of perturbation field and then form and truncate its moment hierarchy by a technique that is compatible with the type of fluctuations assumed. Thus, a two fold purpose would be achieved. First, a model would be constructed which describes a special non-uniformity about which conclusions can be drawn that may be of value to cosmology. Second, and perhaps more significant, would be the demonstration that this procedure does work and is therefore relevant to perturbation problems of general relativity.

In order to best serve the process of demonstrating the relevancy of the moment hierarchy technique to general relativity, a simple nonuniform model is used. It is simple in the sense that only one function is required to describe the perturbation field rather than ten which are needed to treat a general perturbation problem. This random function is also assumed to contain random perturbations which are small compared to the average field. Thus, the previously discussed higher-order-moment-discard approximation will be used for the truncation of the moment hierarchy which will be constructed in a way analogous to fluid turbulence theory. Even for this relatively simple model, the

mathematical manipulations are quite extensive. It will be shown that this particular model is relevant to the study of long wavelength perturbations and deformations of the size which approaches the length characteristic of the universe itself. Thus, the model, though highly restrictive, has a special relevancy and direct applicability to cosmology.

IV. METRIC TENSOR

A. SPECIAL FORM FOR THE MODEL

The process of building the model begins with the formulation of the metric tensor $g_{\mu\nu}$. This can be accomplished by giving first the equation for the square of the line element ds relative to a given coordinate system x^μ and then computing $g_{\mu\nu}(x^\mu)$ by the relation

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \ . \tag{4-1}$$

To this end, consider first the line element for uniform cosmological models. It is called the Robertson-Walker line element and in spherical coordinates is given by [1]

$$ds^2 = (dx^0)^2 - e^{G(x^0,r)}d\sigma^2 \tag{4-2}$$

where

$$d\sigma^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{4-3}$$

and

$$e^{G(x^0,r)} = \frac{R_G^2(x^0)}{r_0^2\left(1 + \frac{k}{4}\frac{r^2}{r_0^2}\right)^2} \ . \tag{4-4}$$

The parameter r_0 is an arbitrary normalization constant with dimensions of length for the radial coordinate r. k is the curvature characteristic which may take only the values -1, 0, and +1. The time coordinate is x^0 and the three spatial coordinates are r, θ, and ϕ. The function $R_G(x^0)$ is called the scale length.

Adler [1] points out that the spatial part of the line element of Eq. (4-2) describes a three dimensional geometry which is homogeneous and isotropic with a uniform radius of curvature R_G. Associated with the three values of the curvature characteristic k are three intrinsically different kinds of uniform geometries. The first, for which k is equal to zero, is the familiar Euclidean geometry. This geometry is based upon the postulate that through any point outside of a straight line, one, and only one parallel line can be drawn. The case where k is equal to +1 corresponds to the uniform geometry first discussed by Riemann which is based on the postulate that any two straight lines in a plane always intersect each other. Finally, the case where k is equal to -1 corresponds to the noneuclidean geometry of Bolyai-Lobachevaki. This uniform geometry is based upon the postulate that through a point outside of a straight line, an unlimited number of straight lines can be drawn which do not intersect the given line and which lie between two straight lines that do intersect the given line at infinity [26].

Now, if the function $G(x^0,r)$ were replaced by an arbitrary function $M(x^0,r,\theta,\phi)$, the resulting line element would no longer be necessarily uniform. In fact, a positive function $R_M(x^0,r,\theta,\phi)$ could be defined such that

$$e^{M(x^0,r,\theta,\phi)} = \frac{R_M^{\,2}(x^0,r,\theta,\phi)}{r_0^{\,2}\left[1+\frac{k}{4}\frac{r^2}{r_0^{\,2}}\right]^2} \quad . \tag{4-5}$$

Thus, at every point x^μ, a tangent Robertson-Walker geometry could be constructed with a radius of curvature R_G equal to the function $R_M(x^\mu)$. Hence, the function R_M could be viewed in this sense as a "local radius of curvature" or "local scale length."

As stated previously, one of the objectives of this dissertation is to demonstrate the applicability of a principle of fluid turbulence to perturbation problems of general relativity by means of a simple model. Clearly, a model whose geometry is described by the line element of Eq. (4-2) with G replaced by M is simple because the entire nonuniformity is described by the single function M (in general, ten independent functions are needed to describe a nonuniform metric $g_{\mu\nu}(x^\mu)$ because $g_{\mu\nu}$ is symmetric with respect to μ,ν). Thus, such a model is a convenient choice for such a demonstration. It is therefore assumed that the line element of the model is given by

$$ds^2 = (dx^0)^2 - e^{M(x^0,r,\theta,\phi)} d\sigma^2 \ . \qquad (4\text{-}6)$$

Because this model is to be described by statistical methods, the value of the function M at any given point must possess a certain amount of randomness. That is, it must be viewed as a stochastic process or random field. Hence, M is a function not only of the four space-time coordinates (x^0,r,θ,ϕ) but also of an ensemble parameter ξ which labels each of those functions M that are possible for a given experiment. Hence, Eq. (4-6) describes not just

one geometry, but an entire family of geometries each of which must obey the Einstein field equations.

The notion of an experiment in the study of stochastic processes is well defined. See, for example, Papoulis [18]. However, for the purposes here, there is an interesting connection between an experiment (as used in statistical theory) and a modern concept called superspace. Wheeler [5] defines superspace as the manifold each of whose "points" represents one three-geometry. A given submanifold of superspace is therefore a history of the geometry undergone by physical space. Hence, the term experiment as used in this paper would indicate an observation of the "state vector" of a possible three-geometry in superspace at a given time. Certainly, the set of possible state vectors for a given experiment is severely limited by Eq. (4-6).

If the tensor form of the line element given by Eq. (4-1) is compared with the line element given by Eq. (4-6) with

$$x^0 = x^0$$

$$x^1 = r$$

$$x^2 = \theta$$

$$x^3 = \phi \;, \qquad (4\text{-}7)$$

it is easily found that the nonvanishing elements of the metric tensor for the model must be

$$g_{00} = 1$$

$$g_{11} = -\,e^M \qquad \begin{matrix}(4\text{-}8)\\ \text{cont.}\end{matrix}$$

$$g_{22} = - r^2 e^M$$

$$g_{33} = - r^2 \sin^2\theta e^M \quad . \tag{4-8}$$

B. DECOMPOSITION OF THE METRIC TENSOR INTO ITS AVERAGE AND FLUCTUATING COMPONENTS.

In fluid turbulence theory, each of the random field variables (pressure and velocity) of the governing Navier-Stokes equation is decomposed into its average and fluctuating components as described in Chapter II. This same decomposition may be performed upon the random metric tensor $g_{\mu\nu}$ which is the field variable of general relativity. That is, $g_{\mu\nu}$ may be written as

$$g_{\mu\nu} = \langle g_{\mu\nu} \rangle + f_{\mu\nu} \tag{4-9}$$

where $\langle g_{\mu\nu} \rangle$ is the average of $g_{\mu\nu}$ and $f_{\mu\nu}$ is its fluctuating component. This average is understood to be an ensemble average. That is, to take an average at each point x^μ, the value of the random field from each member of the ensemble of possible states of the system which occurs with a certain probability are summed and divided by the number of ensemble members. As this number tends towards infinity, this sum tends towards the average value. (The concept of average is discussed in more detail in Appendix B.)

As seen in Eq. (4-8), the randomness of $g_{\mu\nu}$ is contained in the stochastic process e^M. This process can also be decomposed at each point in space-time into two functions such that

$$e^M = \langle e^M \rangle + f_M \tag{4-10}$$

where $\langle e^M \rangle$ is the average of e^M and f_M is the fluctuation of e^M about its average. By Eq. (4-8), the nonvanishing components of $\langle g_{\mu\nu} \rangle$ and $f_{\mu\nu}$ must be

$$\langle g_{00} \rangle = 1$$

$$\langle g_{11} \rangle = - \langle e^M \rangle$$

$$\langle g_{22} \rangle = - r^2 \langle e^M \rangle$$

$$\langle g_{33} \rangle = - r^2 \sin^2\theta \langle e^M \rangle$$

$$f_{11} = - f_M$$

$$f_{22} = - r^2 f_M$$

$$f_{33} = - r^2 \sin^2\theta f_M \quad . \tag{4-11}$$

Another stochastic process $\tau(x^\mu)$ can be defined as the ratio of the fluctuation of e^M to its average. That is,

$$\tau \equiv \frac{f_M}{\langle e^M \rangle} \quad . \tag{4-12}$$

If $f_{\mu\nu}$ is expressed in terms of τ, its nonvanishing elements become

$$f_{11} = - \langle e^M \rangle \tau$$

$$f_{22} = - r^2 \langle e^M \rangle \tau$$

$$f_{33} = - r^2 \sin^2\theta \langle e^M \rangle \tau \quad . \tag{4-13}$$

Because the average of f_M must be zero according to Eq. (4-10), it follows that

$$\langle \tau \rangle = 0 \quad . \tag{4-14}$$

Also by Eq. (4-10), it follows that

$$\tau \geq -1 \qquad (4\text{-}15)$$

because e^M and therefore $<e^M>$ must be positive. Certainly models with small perturbations would be ones for which the ratio of the nonvanishing components of $f_{\mu\nu}$ to $<g_{\mu\nu}>$ is small; that is, for these models, the magnitude of τ would be very small compared to one. As stated in Chapter III, it is this kind of perturbation that will be considered in this work.

1. Requirement for the Average Component of the Metric Tensor

Suppose it is assumed that the average metric tensor $<g_{\mu\nu}>$ is Robertson-Walker. Then, the average geometry would be homogeneous and isotropic everywhere. This assumption, which is used extensively in the literature [7, 14, 24], is convenient because of its relatively simple mathematical form. Furthermore, in the limit of zero fluctuations, the model becomes determinate and the geometry is automatically uniform.

Thus, the nonvanishing components of the metric tensor $<g_{\mu\nu}>$ for the model are given by

$$<g_{00}> = 1$$

$$<g_{11}> = -\, e^{G(x^0,r)}$$

$$<g_{22}> = -\, r^2 e^{G(x^0,r)}$$

$$<g_{33}> = -\, r^2 \sin^2\theta\, e^{G(x^0,r)} \quad . \qquad (4\text{-}16)$$

Because the average metric is assumed to be everywhere homogeneous and isotropic and is given by the above equations, it follows that $\langle e^{M} \rangle = e^{G}$. By virtue of Eqs. (4-4) and (4-5), it can be easily seen that

$$\langle R_M^2(x^0,r,\theta,\phi)\rangle = R_G^2(x^0) \tag{4-17}$$

where $R_M(x^0,r,\theta,\phi)$ is the already discussed local scale length of the random metric and R_G is the universal scale length of the average metric. If $\Delta R^2(x^0,r,\theta,\phi)$ is the fluctuation of R_M^2 from R_G^2 such that

$$R_M^2 = R_G^2 + \Delta R^2 \,, \tag{4-18}$$

then by Eq. (4-12), it follows that

$$\tau = \frac{\Delta R^2}{R_G^2} \quad . \tag{4-19}$$

Thus, τ can be interpreted as the ratio of the fluctuation of the local radius of curvature squared to the average radius of curvature squared.

2. <u>Requirement for the Fluctuating Component of the Metric Tensor</u>

In fluid turbulence theory, a convenient requirement concerning the statistics of the perturbation field is the requirement of stationarity [13]. If this statistical quality is borrowed and applied to the present model, some simplifications can be expected by assuming the statistics of the turbulent field τ to be stationary with respect to a change of coordinates on a spatial hypersurface of constant

x^0. Clearly, the statistics may not be assumed to be stationary with respect to the time coordinate x^0 because this would imply a model universe that is not in a state of dynamic evolution (i.e. expanding). In Appendix C, the implication of this concept is discussed and the important formula given below is derived.

$$\partial_j \langle h[\tau(x^\mu)]\rangle = 0 \tag{4-20}$$

where $j = 1, 2, 3$ and $h(\tau)$ is any function of the stochastic process τ. Clearly this is a very useful formula for by it many bothersome terms are eliminated.

V. EINSTEIN FIELD EQUATIONS FOR THE FLUCTUATING METRIC

A. DESCRIPTION OF THE FIELD EQUATIONS IN GENERAL

Just as the Navier-Stokes equation relates the velocity field to the pressure field in a fluid, so do the Einstein field equations relate the geometry of space to the energy content of that space [1]. Being a tensor theory, the field equations are expressible in a form independent of any coordinate system. The geometrical variable of the field equations has already been identified as the metric tensor $g_{\mu\nu}$. The energy content variable is called the energy momentum tensor $T_{\mu\nu}$ which is developed and discussed later. It should be stated, however, that the field equations for the model itself are not expressed in covariant form because a special coordinate system given by Eq. (4-7) has already been imposed.

Starting with the metric tensor $g_{\mu\nu}$, the field equations can be constructed as follows. (This procedure is given by Adler [1] and is used in the subsequent sections to build the field equations specific to the metric tensor of the proposed model.)

First, the metric tensor $g_{\mu\nu}$ is substituted into the following expression.

$$[\mu\nu,\beta] = \frac{1}{2}\left(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu}\right) \tag{5-1}$$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$. $[\mu\nu,\beta]$ is the Christoffel symbol of the

first kind and is a set of 64 elements which are symmetric with respect to their μ, ν indices.

Next, a quantity $\Gamma^{\alpha}_{\mu\nu}$, the Christoffel symbol of the second kind, which also is a set of 64 elements, is calculated by

$$\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\beta}[\mu\nu,\beta] \tag{5-2}$$

where the identical indices indicate a summation over all β. $\Gamma^{\alpha}_{\mu\nu}$ is also symmetric with respect to its lower indices. The object $g^{\alpha\beta}$, called the second rank contravariant metric tensor, is related to the metric tensor $g_{\mu\nu}$ by

$$g^{\alpha\beta}g_{\mu\beta} = \delta^{\alpha}_{\mu} \quad . \tag{5-3}$$

Next, the Ricci tensor $R_{\mu\nu}$ is formed by

$$R_{\mu\nu} = \partial_{\nu}\Gamma^{\alpha}_{\mu\alpha} - \partial_{\rho}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\alpha}_{\rho\nu}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\alpha}_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} \quad . \tag{5-4}$$

It is a set of 16 elements. The Ricci tensor is then used to calculate the mixed Ricci tensor $R^{\lambda}{}_{\nu}$ given by

$$R^{\lambda}{}_{\nu} = g^{\lambda\delta}R_{\delta\nu} \quad . \tag{5-5}$$

Next, the Ricci scalar R is formed by the contraction of the mixed Ricci tensor $R^{\lambda}{}_{\nu}$ given by

$$R = R^{\lambda}{}_{\lambda} \tag{5-6}$$

Finally, the mixed Ricci tensor of Eq. (5-5) and the Ricci scalar of Eq. (5-6) are used to form the mixed Einstein tensor $G^{\lambda}{}_{\nu}$.

$$G^{\lambda}{}_{\nu} = R^{\lambda}{}_{\nu} - \frac{1}{2}\,\delta^{\lambda}{}_{\nu}R \quad . \tag{5-7}$$

The Einstein field equations are then formed by equating the mixed Einstein tensor $G^{\lambda}{}_{\nu}$ to another tensor $\kappa T^{\lambda}{}_{\nu}$ called the mixed energy momentum tensor, an object which is related to $T_{\mu\nu}$ and is discussed later.

Thus, the mixed form of the Einstein field equations as they are used in this work are

$$\kappa T^{\lambda}{}_{\nu} = G^{\lambda}{}_{\nu} \tag{5-8}$$

where

$$\kappa = -\frac{8\pi G}{c^2} \quad .$$

B. CONSTRUCTION OF THE CHRISTOFFEL SYMBOLS OF THE SECOND KIND FOR THE MODEL

The task of constructing the field equations for the proposed model itself may now begin as outlined in the preceeding section. From the field equations, the moment hierarchy will be determined.

An important object that is immediately needed to form the Christoffel symbols is the second rank contravariant metric tensor $g^{\mu\nu}$. This object can be computed from Eqs. (4-8) and (5-3). The nonvanishing elements of this tensor are

$$g^{00} = 1$$

$$g^{11} = -\,e^{-M}$$

$$g^{22} = \frac{-e^{-M}}{r^2} \tag{5-9}$$

(5-9) cont.

$$g^{33} = \frac{-e^{-M}}{r^2 \sin^2\theta} \qquad \text{cont. (5-9)}$$

The next step is to substitute the metric tensor of Eq. (4-8) into Eq. (5-1) which yields $[\mu\nu,\beta]$, the Christoffel symbol of the first kind. Because the second rank contravariant metric tensor $g^{\mu\nu}$ is diagonal, the calculation of $\Gamma^{\alpha}_{\mu\nu}$, the Christoffel symbol of the second kind, by means of Eq. (5-2) is simplified.

The calculation of $\Gamma^{\alpha}_{\mu\nu}$ is still somewhat lengthy but is entirely straightforward. The nonvanishing components of $\Gamma^{\alpha}_{\mu\nu}$ turn out to be

$$\Gamma^{0}_{11} = \frac{1}{2}\, e^{M} \partial_0 M$$

$$\Gamma^{0}_{22} = \frac{1}{2}\, r^2 e^{M} \partial_0 M$$

$$\Gamma^{0}_{33} = \frac{1}{2}\, r^2 \sin^2\theta\, e^{M} \partial_0 M$$

$$\Gamma^{1}_{01} = \Gamma^{1}_{10} = \frac{1}{2}\, \partial_0 M$$

$$\Gamma^{1}_{11} = \frac{1}{2}\, \partial_r M$$

$$\Gamma^{1}_{21} = \Gamma^{1}_{12} = \frac{1}{2}\, \partial_\theta M$$

$$\Gamma^{1}_{31} = \Gamma^{1}_{13} = \frac{1}{2}\, \partial_\phi M$$

$$\Gamma^{1}_{22} = -\frac{1}{2}\, r^2 (\partial_r M + \frac{2}{r})$$

$$\Gamma^{1}_{33} = -\frac{1}{2}\, r^2 \sin^2\theta (\partial_r M + \frac{2}{r})$$

$$\Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2}\, \partial_0 M \qquad \text{(5-10) cont.}$$

$$\Gamma^2_{11} = -\frac{1}{2}\frac{1}{r^2}\partial_\theta M \qquad \text{cont. (5-10)}$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2}(\partial_r M + \frac{2}{r})$$

$$\Gamma^2_{22} = \frac{1}{2}\partial_\theta M$$

$$\Gamma^2_{32} = \Gamma^2_{23} = \frac{1}{2}\partial_\phi M$$

$$\Gamma^2_{33} = -\frac{1}{2}\sin^2\theta(2\cot\theta + \partial_\theta M)$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2}\partial_0 M$$

$$\Gamma^3_{11} = -\frac{1}{2}\frac{1}{r^2\sin^2\theta}\partial_\phi M$$

$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{2}(\partial_r M + \frac{2}{r})$$

$$\Gamma^3_{22} = -\frac{1}{2}\frac{1}{\sin^2\theta}\partial_\phi M$$

$$\Gamma^3_{32} = \Gamma^3_{23} = \frac{1}{2}(2\cot\theta + \partial_\theta M)$$

$$\Gamma^3_{33} = \frac{1}{2}\partial_\phi M \quad .$$

C. CONSTRUCTION OF THE RICCI TENSOR FOR THE MODEL

The next step is the calculation of the Ricci tensor $R_{\mu\nu}$ from Eqs. (5-4) and (5-10). However, it is anticipated that only the diagonal elements of the Ricci tensor are needed. (This will be justified in the subsequent section.) These diagonal elements are

$$R_{00} = \frac{3}{2}\partial_0^2 M + \frac{3}{4}(\partial_0 M)^2$$

$$R_{11} = \partial_r^2 M + \frac{1}{2}\frac{1}{r^2}\partial_\theta^2 M + \frac{1}{2}\frac{1}{r^2\sin^2\theta}\partial_\phi^2 M \qquad \text{(5-11) cont.}$$

$$- \frac{1}{2} e^{M}\partial_0^2 M - \frac{3}{4} e^{M}(\partial_0 M)^2 + \frac{1}{4}\frac{1}{r^2}(\partial_\theta M)^2$$

$$+ \frac{1}{r}\partial_r M + \frac{1}{2}\frac{\cot\theta}{r^2}\partial_\theta M + \frac{1}{4}\frac{1}{r^2\sin^2\theta}(\partial_\phi M)^2$$

$$R_{22} = - \frac{1}{2} r^2 e^{M}\partial_0^2 M + \frac{1}{2} r^2\partial_r^2 M + \partial_\theta^2 M$$

$$+ \frac{1}{2}\frac{1}{\sin^2\theta}\partial_\phi^2 M + \frac{1}{4} r^2(\partial_r M)^2 + \frac{1}{4}\frac{1}{\sin^2\theta}(\partial_\phi M)^2$$

$$+ \frac{1}{2}\cot\theta\ \partial_\theta M + \frac{3}{2} r\partial_r M - \frac{3}{4} r^2 e^{M}(\partial_0 M)^2$$

$$R_{33} = - \frac{1}{2} r^2\sin^2\theta e^{M}\partial_0^2 M + \frac{1}{2} r^2\sin^2\theta\partial_r^2 M + \frac{1}{2}\sin^2\theta\partial_\theta^2 M$$

$$+ \partial_\phi^2 M - \frac{3}{4} r^2\sin^2\theta e^{M}(\partial_0 M)^2 + \sin^2\theta\ \cot\theta\ \partial_\theta M$$

$$+ \frac{1}{4} r^2\sin^2\theta(\partial_r M)^2 + \frac{1}{4}\sin^2\theta(\partial_\theta M)^2$$

$$+ \frac{3}{2} r\ \sin^2\theta\partial_r M\ . \tag{5-11}$$

D. CONSTRUCTION OF THE DIAGONAL ELEMENTS OF THE MIXED EINSTEIN TENSOR FOR THE MODEL

In order to construct the mixed Einstein tensor, the two objects $R^\lambda{}_\nu$ and R must be found.

The mixed Ricci tensor $R^\lambda{}_\nu$ is easily calculated from Eq. (5-5). Since $g^{\lambda\delta}$ is a diagonal tensor, $R^\lambda{}_\nu$ can be written as

$$R^\lambda{}_\nu = g^{(\lambda)(\lambda)}R_{(\lambda)\nu} \tag{5-11}$$

where the parenthetical indices indicate no summation with

respect to λ. Equation (5-11) justifies the assertion that only the diagonal components of $R_{\mu\nu}$ are needed if only the diagonal elements of $R^{\lambda}{}_{\nu}$ are to be calculated because no off-diagonal mixing occurs by virtue of the diagonality of $g^{\lambda\nu}$.

The Ricci scalar R is easily calculated from the diagonal elements of $R^{\lambda}{}_{\nu}$ by Eq. (5-6). Finally, the components of the mixed Einstein tensor $G^{\lambda}{}_{\nu}$ given by Eq. (5-7) turn out to be

$$G^0{}_0 = -\frac{3}{4}(\partial_0 M)^2 + e^{-M}[\partial_r^2 M + \frac{1}{r^2}\partial_\theta^2 M$$

$$+ \frac{1}{r^2\sin^2\theta}\partial_\phi^2 M + \frac{1}{4}\frac{1}{r^2}(\partial_\theta M)^2 + \frac{2}{r}\partial_r M$$

$$+ \frac{\cot\theta}{r^2}\partial_\theta M + \frac{1}{4}\frac{1}{r^2\sin^2\theta}(\partial_\phi M)^2 + \frac{1}{4}(\partial_r M)^2]$$

$$G^1{}_1 = -\partial_0^2 M - \frac{3}{4}(\partial_0 M)^2 + e^{-M}[\frac{1}{2}\frac{1}{r^2}\partial_\theta^2 M$$

$$+ \frac{1}{2}\frac{1}{r^2\sin^2\theta}\partial_\phi^2 M + \frac{1}{r}\partial_r M + \frac{1}{2}\frac{\cot\theta}{r^2}\partial_\theta M$$

$$+ \frac{1}{4}(\partial_r M)^2]$$

$$G^2{}_2 = -\partial_0^2 M - \frac{3}{4}(\partial_0 M)^2 + e^{-M}[\frac{1}{2}\partial_r^2 M + \frac{1}{2}\frac{1}{r^2\sin^2\theta}\partial_\phi^2 M$$

$$+ \frac{1}{4}\frac{1}{r^2}(\partial_\theta M)^2 + \frac{1}{2}\frac{1}{r}\partial_r M + \frac{1}{2}\frac{\cot\theta}{r^2}\partial_\theta M]$$

$$G^3{}_3 = -\partial_0^2 M - \frac{3}{4}(\partial_0 M)^2 + e^{-M}[\frac{1}{2}\partial_r^2 M + \frac{1}{2}\frac{1}{r^2}\partial_\theta^2 M$$

(5-12)
cont.

$$+ \frac{1}{2}\frac{1}{r}\partial_r M + \frac{1}{4}\frac{1}{r^2 \sin^2\theta}(\partial_\phi M)^2 \quad . \qquad \text{cont. (5-12)}$$

The off-diagonal components are not given because they are not needed. However, calculations similar to the ones required to obtain Eq. (5-12) reveal that the off-diagonal components of $G^\lambda{}_\nu$ are in general non zero and contain mixed partial derivatives of second order and second degree products of first partial derivatives with respect to all four coordinates.

It should be noted that if M is a special function of only the coordinates x^0 and r, namely

$$M = G(x^0, r) \tag{5-13}$$

where G is given by Eq. (4-4), then the line element is identical to the line element of uniform cosmological models [1]. In this case, as should be expected, Eq. (5-12) is in fact identical to the diagonal elements of $G^\lambda{}_\nu$ for these cosmologies.

E. CONSTRUCTION OF THE DYNAMIC EQUATIONS FOR THE MODEL VIA THE EINSTEIN TENSOR

It should be noted that only two "dynamic equations" for $g_{\mu\nu}$ are used to describe uniform cosmological models[1]. But each expression of Eq. (5-12) represents one side of four "dynamic equations." If the proposed model of this study is to pass to a uniform cosmological model for arbitrarily small perturbations, then the number of equations

which should be treated as dynamical equations ought to be two rather than four.

In order to see how to construct two equations from equation set (5-12), it should be observed that $G^0{}_0$ contains second partial derivatives with respect to all four coordinates x^0, r, θ, and ϕ. But each of the remaining objects $G^1{}_1$, $G^2{}_2$, and $G^3{}_3$ contain second partial derivatives with respect to only two of the three possible spatial coordinates. This assymetry can be removed if $G^0{}_0$ is taken as one side of one of the desired two "dynamical equations" and one third the sum $G^j{}_j$ such that

$$G^j{}_j = G^1{}_1 + G^2{}_2 + G^3{}_3 \tag{5-14}$$

is taken as the other. These two equations contain dynamical information concerning $g_{\mu\nu}$ and contain second partial derivatives with respect to all coordinates. They are also found to degenerate into the usual two cosmological equations when G is replaced by M [1].

By virtue of Eq. (5-8), the two dynamic equations for the model are

$$\kappa T^0{}_0 = G^0{}_0 \tag{5-15}$$

and

$$\frac{\kappa}{3} T^j{}_j = \frac{G^j{}_j}{3} \ . \tag{5-16}$$

From Eq. (5-12), $G^j{}_j$ is given by

$$G^j{}_j = -\,3\partial_0^2 M - \frac{9}{4}(\partial_0 M)^2 + e^{-M}[\partial_r^2 M + \frac{1}{r^2}\,\partial_\theta^2 M \qquad \text{(5-17) cont.}$$

$$+ \frac{1}{r^2 \sin^2\theta} (\partial_\phi^2 M) + \frac{1}{4} (\partial_r M)^2 + \frac{1}{4} \frac{1}{r^2} (\partial_\theta M)^2 \qquad \text{cont.} \tag{5-17}$$

$$+ \frac{1}{4} \frac{1}{r^2 \sin^2\theta} (\partial_\phi M)^2 + \frac{2}{r} (\partial_r M) + \frac{\cot\theta}{r^2} \partial_\theta M].$$

For convenience, $G^0{}_0$ can be expressed in terms of $G^j{}_j$.

$$G^0{}_0 = G^j{}_j + 3\partial_0^2 M + \frac{3}{2} (\partial_0 M)^2 \quad . \tag{5-18}$$

F. CONSTRUCTION OF THE ENERGY-MOMENTUM TENSOR FOR THE MODEL

To proceed further, it is necessary to express the energy-momentum tensor in a form that is specific to the metric used for this model. It is to be understood that the mixed energy-momentum tensor $T^\mu{}_\nu$ is also a stochastic process because it depends, via the Einstein field equations, upon the metric tensor which is a stochastic process. The general form of the energy-momentum tensor is known to depend upon three quantities, $\bar{D}$ the proper mass density, $\bar{P}$ the proper pressure, and u the velocity of the medium. These quantities $\bar{D}$ and $\bar{P}$ are termed "proper" because they are referred to the proper frame which is that frame that is instantaneously moving with the medium. From any other frame, the well known effects upon the mass and volume due to the relative velocity u must be taken into account.

The representation of the general energy-momentum tensor with respect to the metric that is used in this model is derived in Appendix D. All of the elements of this tensor are given by Eq. (D-19). The two related objects which are important here are $T^0{}_0$ and $T^j{}_j$;

$$T^0{}_0 = D + \frac{P}{c^2} u^2$$

$$T^j{}_j = - Du^2 - \frac{P}{c^2} (3 - 2u^2) \tag{5-19}$$

where

$$D = \text{proper density } \overline{D} \times \frac{1}{1-u^2}$$

$$P = \text{proper pressure } \overline{P} \times \frac{1}{1-u^2} \tag{5-20}$$

u = local velocity of the matter in units of c.

Because of the recent discovery of a possible 3^0K black-body background radiation [6, 23], a relevant energy-momentum tensor is one which describes disordered electromagnetic radiation of this type. However, this radiation is believed to be no longer important to the dynamics of the universe (because now the matter density far exceeds the radiation density) although it is believed to have been in the more distant past. The energy-momentum tensor which is relevant to the current stage of expansion is one which describes the random motion of matter in the form of galaxies and clusters of galaxies. Eq. (D-4) of Appendix D gives for these two cases

$$\overline{P} = \alpha c^2 \overline{D} \tag{5-21}$$

where $\alpha = 0$ if universe is matter dominated and

$\alpha = 1/3$ if universe is radiation dominated.

Hence, by Eq. (5-20),

$$P = \alpha c^2 D. \tag{5-22}$$

VI. MOMENT EQUATION HIERARCHY

A. AVERAGE OF THE DYNAMIC EQUATIONS

Equations (5-15) and (5-16) are the "equations of motion" for the assumed metric tensor of Eq. (4-8) and the energy-momentum tensor appropriate for that metric. These equations should be viewed as the analogue of the Navier-Stokes equation for $\vec{V}$ and P given by Eq. (A-1) because, as yet, the decomposed metric tensor of Eq. (4-9) has not been inserted into them.

However, the substitution of the decomposed metric can be accomplished rather easily. A quick inspection of $G^j{}_j$ and $G^0{}_0$ reveals that they are made up exclusively of partial derivatives of the stochastic process M and a factor e^{-M}. From Eqs. (4-10) and (4-12), it is easily found that

$$e^M = e^G(1 + \tau) \qquad (6\text{-}1)$$

where e^G is associated with the average metric and τ with the fluctuating part of the metric.

Equation (6-1) can be used to separate $G^j{}_j$ and $G^0{}_0$ into two parts, one which contains only the nonrandom function G and the other which contains the random function τ in combinations of G and τ. The part which is made up only of G terms must be $G^j{}_j$ and $G^0{}_0$ based on the average metric (because this part cannot depend on τ) while the remaining part must be the source terms (because these terms must depend on τ). The resulting expressions for $G^0{}_0$ and $G^j{}_j$

are (using Eq. (4-4) which is the definition of e^G and with R equal to R_G and Eqs. (5-15) and (5-16))

$$\kappa T^0{}_0 = G^0{}_0 = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} - \frac{3R'}{R}\left(\frac{\partial_0\tau}{1+\tau}\right) - \frac{3}{4}\frac{(\partial_0\tau)^2}{(1+\tau)^2}$$

$$+ \frac{e^{-G}}{1+\tau}\left\{\left[\frac{\partial_r^2\tau}{1+\tau} - \frac{3}{4}\frac{(\partial_r\tau)^2}{(1+\tau)^2}\right]\right.$$

$$+ \frac{1}{r^2}\left[\frac{\partial_\theta^2\tau}{1+\tau} - \frac{3}{4}\frac{(\partial_\theta\tau)^2}{(1+\tau)^2}\right]$$

$$\left.+ \frac{1}{r^2\sin^2\theta}\left[\frac{\partial_\phi^2\tau}{1+\tau} - \frac{3}{4}\frac{(\partial_\phi\tau)^2}{(1+\tau)^2}\right]\right\}$$

$$+ \frac{\cot\theta}{r^2}\left(\frac{\partial_\theta\tau}{1+\tau}\right) + \frac{2}{r}\frac{1}{1+\dfrac{kr^2}{4r_0^2}}\left(\frac{\partial_r\tau}{1+\tau}\right)$$

$$+ \frac{3k}{R^2}\frac{\tau}{1+\tau} \tag{6-2}$$

$$\kappa T^j{}_j = G^j{}_j = G^0{}_0 - \frac{6R''}{R} - \frac{3\partial_0^2\tau}{1+\tau} + \frac{3}{2}\frac{(\partial_0\tau)^2}{(1+\tau)^2}$$

$$- \frac{6R'}{R}\left(\frac{\partial_0\tau}{1+\tau}\right) . \tag{6-3}$$

If these equations are averaged, the result is the desired dynamic equations for the average variables $\langle g_{\mu\nu}\rangle$ and $\langle T^0{}_0\rangle$.

These averaged equations can be immediately simplified by using the assumption of spatial stationarity that is discussed in Ch. IV. Thus, all averages of the form $\partial_j\langle h(\tau)\rangle$ vanish by virtue of Eq. (4-20). Therefore, in

particular,

$$<\partial_j\left(\frac{1}{1+\tau}\right)> = 0 \ . \tag{6-4}$$

From this follow the identities

$$<\left[\frac{2(\partial_j\tau)^2}{(1+\tau)^3}\right]> = <\left[\frac{\partial_j^2\tau}{(1+\tau)^2}\right]> \tag{6-5}$$

and

$$<\left[\frac{\partial_j\tau}{(1+\tau)^2}\right]> = 0 \quad . \tag{6-6}$$

Thus, with Eqs. (6-5) and (6-6), the average of Eqs. (6-2) and (6-3) are

$$\kappa<T^0{}_0> + \frac{3R'}{R}<\left(\frac{\partial_0\tau}{1+\tau}\right)> + \frac{3}{4}<\left[\frac{(\partial_0\tau)^2}{(1+\tau)^2}\right]>$$

$$- \frac{5}{4}e^{-G}\left\{<\left[\frac{(\partial_r\tau)^2}{(1+\tau)^3}\right]> + \frac{1}{r^2}<\left[\frac{(\partial_\theta\tau)^2}{(1+\tau)^3}\right]>\right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}<\left[\frac{(\partial_\phi\tau)^2}{(1+\tau)^3}\right]>\right\} - \frac{3k}{R^2}<\left(\frac{\tau}{1+\tau}\right)>$$

$$= -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{6-7}$$

$$\frac{\kappa}{3}<T^j{}_j> + \frac{3R'}{R}<\left[\frac{\partial_0\tau}{1+\tau}\right]> - \frac{1}{4}<\left[\frac{(\partial_0\tau)^2}{(1+\tau)^2}\right]>$$

$$+ <\left[\frac{(\partial_0^2\tau)}{(1+\tau)}\right]> - \frac{5}{12}e^{-G}\left\{<\left[\frac{(\partial_r\tau)^2}{(1+\tau)^3}\right]>\right.$$

$$\left. + \frac{1}{r^2}<\left[\frac{(\partial_\theta\tau)^2}{(1+\tau)^3}\right]> + \frac{1}{r^2\sin^2\theta}<\left[\frac{(\partial_\phi\tau)^2}{(1+\tau)^3}\right]>\right\} \qquad \text{(6-8) cont.}$$

$$- \frac{k}{R^2} < \left(\frac{\tau}{1+\tau} \right) > = - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \tag{6-8}$$

If Eqs. (5-19) which give expressions for $T^0{}_0$ and $T^j{}_j$ are used, Eqs. (6-7) and (6-8) become (with the aid of Eq. (5-22))

$$\kappa < D(1 + \alpha u^2) > + \frac{3R'}{R} < \left(\frac{\partial_0 \tau}{1+\tau} \right) > + \frac{3}{4} < \left[\frac{(\partial_0 \tau)^2}{(1+\tau)^2} \right] >$$

$$- \frac{5}{4} e^{-G} \left\{ < \left[\frac{(\partial_r \tau)^2}{(1+\tau)^3} \right] > + \frac{1}{r^2} < \left[\frac{(\partial_\theta \tau)^2}{(1+\tau)^3} \right] > \right.$$

$$\left. + \frac{1}{r^2 \sin^2\theta} < \left[\frac{(\partial_\phi \tau)^2}{(1+\tau)^3} \right] > \right\} - \frac{3k}{R^2} < \left(\frac{\tau}{1+\tau} \right) >$$

$$= - \frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{6-9}$$

$$\frac{\kappa}{3} < Du^2(2\alpha - 1) - 3\alpha D > + \frac{3R'}{R} < \left[\frac{(\partial_0 \tau)}{(1+\tau)} \right] > - \frac{1}{4} < \left[\frac{(\partial_0 \tau)^2}{(1+\tau)^2} \right] >$$

$$+ < \left[\frac{(\partial_0^2 \tau)}{(1+\tau)} \right] > - \frac{5}{12} e^{-G} \left\{ < \left[\frac{(\partial_r \tau)^2}{(1+\tau)^3} \right] > \right.$$

$$\left. + \frac{1}{r^2} < \left[\frac{(\partial_\theta \tau)^2}{(1+\tau)^3} \right] > + \frac{1}{r^2 \sin^2\theta} < \left[\frac{(\partial_\phi \tau)^2}{(1+\tau)^3} \right] > \right\}$$

$$- \frac{k}{R^2} < \left(\frac{\tau}{1+\tau} \right) > = - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} . \tag{6-10}$$

These two equations are the desired dynamic equations.

B. CONSTRUCTION OF THE SPECIAL MOMENT EQUATIONS

As stated in Ch. II , it is necessary to determine the functional form of the extra source terms of Eqs. (6-9) and (6-10) in order to solve for the average quantities R, $<T^{0}{}_{0}>$, and $<T^{j}{}_{j}>$.

The basic idea is to construct additional equations for the unknown correlations. This is done by first multiplying the unaveraged Eqs. (6-2) and (6-3) or combinations of these by some "special functions" which involve powers or derivatives of τ and then averaging them. As might be expected, the nonlinearities of the unaveraged equations cause more unknown correlations to be formed which requires the formation of more equations. Although this process can be carried out ad infinitum, a special truncation of this process is required which imposes an additional restriction upon the model. The truncation that will be used is the higher-order-moment discard approximation that is discussed in Ch. III in connection with the fluid turbulence hierarchy. This means that the higher order moment that results from the nonlinearity is discarded because of smallness. Hence, only the first "generation" of the moment hierarchy is needed and it turns out that three special equations are sufficient (see Ch. VII).

The two special combinations of the unaveraged dynamic Eqs. (6-2) and (6-3) which are used as the basis for developing these three equations are

$$\kappa(T^0{}_0 + T^j{}_j) = G^0{}_0 + G^j{}_j$$

$$= 2G^0{}_0 - \frac{6R''}{R} - \frac{3\partial_0^2\tau}{1+\tau} + \frac{3}{2}\frac{(\partial_0\tau)^2}{(1+\tau)^2} - \frac{6R'}{R}\left(\frac{\partial_0\tau}{1+\tau}\right) \tag{6-11}$$

and

$$\kappa(T^0{}_0 - T^j{}_j) = G^0{}_0 - G^j{}_j$$

$$= \frac{6R''}{R} + \frac{3\partial_0^2\tau}{1+\tau} - \frac{3}{2}\frac{(\partial_0\tau)^2}{(1+\tau)^2} + \frac{6R'}{R}\left[\frac{\partial_0\tau}{1+\tau}\right] . \tag{6-12}$$

If Eq. (6-11) is multiplied by the "special function" $1+\tau$ and averaged, the first moment equation is formed. The resulting equation can be simplified by again using Eq. (4-20) which implies that

$$\partial_j^2\langle h(\tau)\rangle = 0 .$$

If $h(\tau)$ is taken to be $\ln(1+\tau)$, then the following identity results

$$\left\langle\left[\frac{\partial_j^2\tau}{1+\tau}\right]\right\rangle = \left\langle\left[\frac{(\partial_j\tau)^2}{(1+\tau)^2}\right]\right\rangle . \tag{6-13}$$

Thus, the simplified first moment equation is

$$\kappa\langle(T^0{}_0 + T^j{}_j)(1+\tau)\rangle - \frac{1}{2}e^{-G}\left\{\left\langle\left[\frac{(\partial_r\tau)^2}{(1+\tau)^2}\right]\right\rangle + \frac{1}{r^2}\left\langle\left[\frac{(\partial_\theta\tau)^2}{(1+\tau)^2}\right]\right\rangle + \frac{1}{r^2\sin^2\theta}\left\langle\left[\frac{(\partial_\phi\tau)^2}{(1+\tau)^2}\right]\right\rangle\right\}$$

$$= -\frac{6k}{R^2} - \frac{6R'^2}{R^2} - \frac{6R''}{R} . \tag{6-14}$$

If Eq. (6-12) is multiplied by τ and then $\partial_0\tau$, the second and third moment equations are produced.

$$\kappa\langle(T^0{}_0 - T^j{}_j)\tau\rangle - 3\left\langle\left[\frac{(\partial_0^2\tau)(\tau)}{(1+\tau)}\right]\right\rangle + \frac{3}{2}\left\langle\left[\frac{(\partial_0\tau)^2(\tau)}{(1+\tau)^2}\right]\right\rangle$$

$$- \frac{6R'}{R}\left\langle\left[\frac{(\partial_0\tau)(\tau)}{(1+\tau)}\right]\right\rangle = 0 \qquad (6\text{-}15)$$

$$\kappa\langle(T^0{}_0 - T^j{}_j)(\partial_0\tau)\rangle - 3\left\langle\left[\frac{(\partial_0^2\tau)(\partial_0\tau)}{(1+\tau)}\right]\right\rangle + \frac{3}{2}\left\langle\left[\frac{(\partial_0\tau)^3}{(1+\tau)^2}\right]\right\rangle$$

$$- \frac{6R'}{R}\left\langle\left[\frac{(\partial_0\tau)^2}{(1+\tau)}\right]\right\rangle = 0\ . \qquad (6\text{-}16)$$

Again, using equation set (5-19), these three equations become

$$\kappa\langle D(1-u^2)(1-3\alpha)(1+\tau)\rangle - \frac{1}{2}\,e^{-G}\left\{\left\langle\left[\frac{(\partial_r\tau)^2}{(1+\tau)^2}\right]\right\rangle\right.$$

$$\left. + \frac{1}{r^2}\left\langle\left[\frac{(\partial_\theta\tau)^2}{(1+\tau)^2}\right]\right\rangle + \frac{1}{r^2\sin^2\theta}\left\langle\left[\frac{(\partial_\phi\tau)^2}{(1+\tau)^2}\right]\right\rangle\right\}$$

$$= -\frac{6k}{R^2} - \frac{6R'^2}{R^2} - \frac{6R''}{R} \qquad (6\text{-}17)$$

$$\kappa\langle[D(1+3\alpha) + Du^2(1-\alpha)]\tau\rangle - 3\left\langle\left[\frac{(\partial_0^2\tau)(\tau)}{(1+\tau)}\right]\right\rangle$$

$$+ \frac{3}{2}\left\langle\left[\frac{(\partial_0\tau)^2(\tau)}{(1+\tau)^2}\right]\right\rangle - \frac{6R'}{R}\left\langle\left[\frac{(\partial_0\tau)(\tau)}{(1+\tau)}\right]\right\rangle = 0 \quad (6\text{-}18)$$

$$\kappa\langle[D(1 + 3\alpha) + Du^2(1 - \alpha)](\partial_0\tau)\rangle - 3\left\langle\left[\frac{(\partial_0^2\tau)(\partial_0\tau)}{(1+\tau)}\right]\right\rangle$$

$$+ \frac{3}{2}\left\langle\left[\frac{(\partial_0\tau)^3}{(1+\tau)^2}\right]\right\rangle - \frac{6R'}{R}\left\langle\left[\frac{(\partial_0\tau)^2}{(1+\tau)}\right]\right\rangle = 0. \qquad (6\text{-}19)$$

VII. TRUNCATION OF THE MOMENT EQUATION HIERARCHY

In both Eqs. (6-9) and (6-10) it can be seen that the source terms contain $(1+\tau)^n$ in their denominators. Because $(1+\tau)^n$ can be expanded in a power series in τ, the source terms can be expressed as an infinite sum of moments of various orders in τ and its derivatives. As mentioned before, an equation for each of these moments can be generated by multiplying the unaveraged Eqs. (6-2) and (6-3) by an appropriate quantity and averaging. The nonlinearities of these equations create a dependence of the desired moment on higher order moments. Because this occurs for all moments, a method is needed to truncate the system in such a way that only a finite number of moments and required equations in the hierarchy are left. In Ch. III three possible truncation schemes are presented. In this work, the higher-order-moment discard method is chosen by assuming that most of the probability mass of the random variable $\tau(x^\mu,\xi)$ is concentrated around $\tau = 0$, so that the higher order moments can be neglected. This obviously limits the applicability of the theory to models of small nonuniformity for which $|\tau| \ll 1$ (see Ch. IV-B).

However, care must be exercised in affecting this truncation. It is not correct to just replace $(1+\tau)^n$ by one in each of the denominators of the above mentioned equations just because this is the first term of the expansion. This

can be seen by an example. Consider the typical term $<\left(\frac{\partial_0 \tau}{1+\tau}\right)>$. This term can be expanded as follows

$$<\left(\frac{\partial_0 \tau}{1+\tau}\right)> = <\partial_0 \tau> - <(\partial_0 \tau)(\tau)> + \ldots \qquad (7\text{-}1)$$

Since $<\partial_0 \tau> = \partial_0 <\tau> = 0$,

$$<\left(\frac{\partial_0 \tau}{1+\tau}\right)> \approx - <(\partial_0 \tau)(\tau)> . \qquad (7\text{-}2)$$

However, if $(1+\tau)$ were falsely replaced by one, the following incorrect conclusion would result which disagrees with the correct Eq. (7-2)

$$<\left(\frac{\partial_0 \tau}{1+\tau}\right)> \approx <\partial_0 \tau> . \qquad (7\text{-}3)$$

Hence, an expansion must be made for every term and that lowest order moment term which does not vanish by virtue of having a zero average is assumed to be a "good" approximation to the term itself.

With this "weak fluctuation" assumption, Eqs. (6-9) (6-10), (6-17), (6-18), and (6-19) become

$$\kappa<D(1+\alpha u^2)> - \frac{3}{2}\frac{R'}{R}\,\partial_0<\tau^2> + \frac{3}{4}<(\partial_0 \tau)^2>$$

$$- \frac{5}{4}\,e^{-G}\left\{<(\partial_r \tau)^2> + \frac{1}{r^2}<(\partial_\theta \tau)^2> + \frac{1}{r^2 \sin^2\theta}<(\partial_\phi \tau)^2>\right\}$$

$$+ \frac{3k}{R^2}<\tau^2> = - \frac{3k}{R^2} - \frac{3R'^2}{R^2} . \qquad (7\text{-}4)$$

$$\frac{\kappa}{3}<Du^2(2\alpha-1) - 3\alpha D> - \frac{3}{2}\frac{R'}{R}\,\partial_0<\tau^2> - \frac{1}{2}\,\partial_0^2<\tau^2> \qquad (7\text{-}5)$$

cont.

$$+ \frac{3}{4} <(\partial_0\tau)^2> - \frac{5}{12} e^{-G}\Big\{<(\partial_r\tau)^2> + \frac{1}{r^2} <(\partial_\theta\tau)^2>$$

$$+ \frac{1}{r^2\sin^2\theta} <(\partial_\phi\tau)^2>\Big\} + \frac{k}{R^2} <\tau^2>$$

$$= - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \qquad \text{cont. (7-5)}$$

$$- \frac{5}{4} e^{-G}\left\{<(\partial_r\tau)^2> + \frac{1}{r^2} <(\partial_\theta\tau)^2> + \frac{1}{r^2\sin^2\theta} <(\partial_\phi\tau)^2>\right\}$$

$$+ \frac{5}{2} \kappa<D(1-u^2)(1-3\alpha)(1+\tau)$$

$$= \frac{-15k}{R^2} - \frac{15R'^2}{R^2} - \frac{15R''}{R} \qquad (7\text{-}6)$$

$$\kappa<[D(1+3\alpha) + Du^2(1-\alpha)]\tau> - \frac{3}{2} \partial_0^2<\tau^2>$$

$$- \frac{3R'}{R} \partial_0<\tau^2> = - 3<(\partial_0\tau)^2> \qquad (7\text{-}7)$$

$$\partial_0<(\partial_0\tau)^2> + \frac{4R'}{R} <(\partial_0\tau)^2>$$

$$= \frac{2}{3} \kappa<[D(1+3\alpha) + Du^2(1-\alpha)]\partial_0\tau> \quad . \qquad (7\text{-}8)$$

In the limit of zero fluctuations, Eqs. (7-4) and (7-5) become the usual two cosmological equations for ρ and R that are given by Adler [1]. Thus, it is natural to look upon these equations as dynamic equations for ρ and R. The source terms which appear in these equations contain the three functions $<\tau^2>$, $<(\partial_0\tau)^2>$, and $\{<(\partial_r\tau)^2> + \frac{1}{r^2} <(\partial_\theta\tau)^2> + \frac{1}{r^2\sin^2\theta} <(\partial_\phi\tau)^2>\}$. Thus, only three

equations are needed to determine these variables which justifies the statement made in Part B of the preceeding chapter that only three equations of the hierarchy need to be calculated. These three equations are Eqs. (7-6), (7-7), and (7-8). Although the above functions $<\tau^2>$, $<(\partial_0\tau)^2>$, and $\{<(\partial_r\tau)^2> + \frac{1}{r^2} <(\partial_\theta\tau)^2> + \frac{1}{r^2\sin^2\theta} <(\partial_\phi\tau)^2>\}$ appear in these equations, additional moments also occur which relate the density D to τ and D to $\partial_0\tau$. It is shown in the succeeding chapters that these moments can be related to the existing moment $<\tau^2>$. Thus, Eqs. (7-4) to (7-8) are soluble among themselves. Because Eqs. (7-6) to (7-8) govern the dependence of the source moments on R, they are called growth equations to convey the idea that they determine the development or growth of a perturbation τ with increasing R.

VIII. REVIEW OF THE RESTRICTIVE ASSUMPTIONS

Before proceeding further, it might be helpful to review the restrictive assumptions that are used to obtain Eqs. (7-4) through (7-8). The first is the very restrictive form of the metric tensor. This form excludes many types of gravitational wave phenomena. All that is seen by an observer would be a local "expansion" or "contraction" of space because, as stated in Ch. IV, it would always be possible to construct an expanding or contracting Robertson-Walker geometry that is "tangent" to the given space at any given point in space-time. The exclusion of such waves must certainly exclude any hope of using this model to describe, in a realistic way, the violent, multiply connected space that Wheeler envisions must typify the quantum fluctuations of space-time.

The second and third restrictions are the restrictions concerning the Robertson-Walker form of the average metric and the spatial stationarity of the fluctuating part. These two assumptions rely more on intuition than upon scientific rigor. They imply that there exists some sort of spatial uniformity in the statistics of the assumed fluctuating geometry. However, it is to be understood that a "new" cosmological principle is not being seriously proposed here by the use of these two assumptions. That these assumptions are used at all reflects a need to simplify the mathematics

of the moment equations in some reasonable way by appealing to some plausible symmetry arguments. Thus, these two restrictions should be viewed only as steps in this particular problem whose purpose is to demonstrate a technique.

Finally, the fourth restriction is the assumption that the higher order moments are negligible owing to an assumed smallness in the anisotropy and inhomogeneity. This assumption is a restriction upon the applicability of the model. That is, in those solution domains for which the magnitude of τ is comparable or larger than one, the use of the model is unwarrantable. If large amplitude fluctuations are to be considered, then perhaps a truncation based upon a phenomenological or a cumulant-discard approximation would be useful. However, in the case of the latter, it might be assumed that the probability distribution of the random variable τ is normal. Then, the higher order moments can be shown to be products of the lower order ones [18]. Such an assumption is made in fluid turbulence theory [3]. But, in the case of this model, τ must be greater than or equal to minus one as given by Eq. (4-15). This creates difficulty with a normal distribution where the probability of τ being +T is equal to the probability of it being -T where T is an arbitrary positive number.

IX. SPECIAL UNIVERSE MODELS

From this point on it is convenient to consider two types of models which correspond to two values of α. The first, for which α is equal to one third, corresponds to a universe made out of electromagnetic radiation while the second corresponds to a universe which consists of cold matter for which α is zero. (α is equal to $\frac{P}{c^2 D}$. See Eq. (5-22).)

It is not practical to treat these models simultaneously by leaving α arbitrary. The cold dust model can be solved exactly owing to the fact that all matter moves freely along geodesics without being accelerated by any pressure gradients (pressure is zero everywhere). But the radiation model does involve pressure gradients which do not allow the elements of the radiation fluid to move freely in space [9]. Therefore, these two types of models are sufficiently different from one another to merit separate consideration.

A. RADIATION FILLED UNIVERSE MODEL

For the radiation filled universe, Eqs. (7-4), (7-5), (7-6), (7-7), and (7-8) become

$$\kappa\rho - \frac{3}{2}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle - \frac{5}{4}e^{-G}\Big\{\langle(\partial_r\tau)^2\rangle + \frac{1}{r^2}\langle(\partial_\theta\tau)^2\rangle + \frac{1}{r^2\sin^2\theta}\langle(\partial_\phi\tau)^2\rangle\Big\} \qquad \text{(9-1) cont.}$$

$$+\frac{3k}{R^2}<\tau^2> = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \qquad \text{cont. (9-1)}$$

$$-\frac{\kappa\rho}{3} - \frac{3}{2}\frac{R'}{R}\partial_0<\tau^2> - \frac{1}{2}\partial_0^2<\tau^2> + \frac{3}{4}<(\partial_0\tau)^2>$$

$$-\frac{5}{12}e^{-G}\left\{<(\partial_r\tau)^2> + \frac{1}{r^2}<(\partial_\theta\tau)^2>\right.$$

$$\left.+\frac{1}{r^2\sin^2\theta}<(\partial_\phi\tau)^2>\right\} + \frac{k}{R^2}<\tau^2>$$

$$= -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \qquad (9\text{-}2)$$

$$-\frac{5}{4}e^{-G}\left\{<(\partial_r\tau)^2> + \frac{1}{r^2}<(\partial_\theta\tau)^2> + \frac{1}{r^2\sin^2\theta}<(\partial_\phi\tau)^2>\right\}$$

$$= \frac{-15k}{R^2} - \frac{15R'^2}{R^2} - \frac{15R''}{R} \qquad (9\text{-}3)$$

$$2\kappa<\tau d> - \frac{3}{2}\partial_0^2<\tau^2> - \frac{3R'}{R}\partial_0<\tau^2>$$

$$= -3<(\partial_0\tau)^2> \qquad (9\text{-}4)$$

$$\partial_0<(\partial_0\tau)^2> + \frac{4R'}{R}<(\partial_0\tau)^2> = \frac{4}{3}\kappa<(\partial_0\tau)d> \qquad (9\text{-}5)$$

where

$$\rho \equiv <D(1 + \frac{u^2}{3})> \qquad (9\text{-}6)$$

and

$$d \equiv D(1 + \frac{u^2}{3}) - \rho \qquad (9\text{-}7)$$

It should be noted that ρ is equal to the average of $T^0{}_0$. Adler [1] points out that $T^0{}_0$ can be interpreted as the total mass density with respect to the given coordinate

system and therefore includes relativistic corrections to the proper mass density.

1. Elimination of the Spatial Derivatives

Now consider the terms which occur in Eqs. (9-1) and (9-2) of the form $<(\partial_\mu \tau)^2>$. It should be noted that only ∂_0 and ∂_r are partial derivatives with respect to a length (in the time and r directions). The remaining partial derivatives ∂_θ and ∂_ϕ are partial derivatives with respect to the coordinates θ and ϕ which are not lengths in themselves. An inspection of the line element of Eq. (4-2) suggests that ∂_0, $e^{-G/2}\partial_r$, $e^{-G/2}\, 1/r\, \partial_\theta$, and $e^{-G/2}\, 1/r\sin\theta\, \partial_\phi$, all of which have dimensions of inverse lengths, play the same roles in the x^0,r,θ, and ϕ directions respectively. Because disturbances of the metric propagate with the speed of light [11, 32], it would be expected that the characteristic inverse lengths associated with these quantities are comparable in magnitude. Thus, it seems reasonable to expect that the spatial terms

$$e^{-G}<(\partial_r \tau)^2>,\ \frac{1}{r^2}\, e^{-G}<(\partial_\theta \tau)^2>,\ \frac{e^{-G}}{(r\ \sin\theta)^2}<(\partial_\phi \tau)^2>$$

could be expressed as mere repetitions of the temporal term $<(\partial_0 \tau)^2>$ multiplied by a coefficient of order one.

Such a fact would simplify Eqs. (9-1) and (9-2) because the terms involving the spatial pertial derivatives would be changed into the already existing terms involving the time derivative.

This consolidation of the terms is performed in Appendix F. Equation (9-3) is used to eliminate the spatial partial derivative terms from Eqs. (9-1) and (9-2). An appropriate renormalization is then performed to bring the right hand sides of the resulting equations back to "canonical form" as they appear in the right hand sides of Eqs. (9-1) and (9-2). Equation (9-4) is used to change the form of the "renormalized" equations into a more convenient form. This last step is not part of the consolidation itself since it just allows the resulting equations to be presented in a more desirable form. The transformed pair of Eqs. (9-1) and (9-2) which are void of spatial partial derivatives are

$$\kappa\rho + \frac{5}{4}\kappa\langle\tau d\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle + \frac{3}{8}\frac{R'}{R}\partial_0\langle\tau^2\rangle - \frac{3}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{9-8}$$

$$-\frac{\kappa\rho}{3} - \frac{1}{4}\kappa\langle\tau d\rangle - \frac{1}{4}\langle(\partial_0\tau)^2\rangle + \frac{1}{8}\frac{R'}{R}\partial_0\langle\tau^2\rangle - \frac{1}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \quad . \tag{9-9}$$

2. Inclusion of the Dynamics for the Radiation Field

Because Eq. (9-3) has been used to remove the unknown

$$e^{-G}\left\{\langle(\partial_r\tau)^2\rangle + \frac{1}{r^2}\langle(\partial_\theta\tau)^2\rangle + \frac{1}{r^2\sin^2\theta}\langle(\partial_\phi\tau)^2\rangle\right\}$$

from Eqs. (9-1) and (9-2), there exists now only four

equations in the system namely, Eqs. (9-4), (9-5), (9-8), and (9-9). But this system of equations is insoluble because they represent four coupled nonlinear differential equations in six unknown functions of the time coordinate x^0. These unknown functions are ρ, R, $\langle\tau^2\rangle$, $\langle(\partial_0\tau)^2\rangle$, $\langle\tau d\rangle$, and $\langle(\partial_0\tau)d\rangle$.

However, there is a way to express $\langle\tau d\rangle$ and $\langle(\partial_0\tau)d\rangle$ in terms of the other four unknown functions. It is not unreasonable to expect that there should exist some relationship between the density fluctuation $d(x^\mu)$ and geometrical fluctuation $\tau(x^\mu)$ because in some sense the geometry fluctuations "drive" the density fluctuations. But it must be remembered that the radiation fluid possesses internal pressure forces which push the fluid elements about as well as do work against any expansions or contractions of space which might occur with amplitude τ [9]. Thus, the problem of determining this relationship between d and τ is in general very complicated. The correct way to do this is to use the equations of motion for the radiation fluid which are given in tensor form by [1]

$$T^{\mu\nu};\ \nu = 0 \tag{9-10}$$

where

$$T^{\mu\nu};\ \nu = \partial_\nu T^{\mu\nu} + \Gamma^{\nu}_{\nu\alpha}\, T^{\mu\alpha} + \Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} \ .$$

Indeed, the relationship between d and τ is approximated in Appendix I via these equations and is expressed in terms of the correlations $\langle\tau d\rangle$ and $\langle(\partial_0\tau)d\rangle$.

An instructive yet simpler way to determine the relationship between d and τ is the following. Consider a small 3-volume element V at rest with respect to the coordinates (r, θ, ϕ). Let M be the total relativistic mass content of this 3-volume. The total relativistic mass density D is thus given by

$$D = \frac{M}{V} \quad . \tag{9-11}$$

A small change ΔD of D can be in general a result of a small change ΔM of M and a small change ΔV of V. That is,

$$\Delta D = \frac{\Delta M}{V} - \frac{M\Delta V}{V^2} \quad . \tag{9-12}$$

Now the quantity $\sqrt{-g}\ d^4x$ is known to be a scalar invariant where g is the determinent of the matrix $g_{\mu\nu}$ [1, 32]. In particular, from special relativity, this quantity is the product of the proper time interval and the proper volume. Because x^0 has already been identified as the time coordinate (see Ch. IV), the volume V must be

$$V = \sqrt{-g}\ d^3x \ . \tag{9-13}$$

In terms of the coordinates r, θ, ϕ,

$$V = \sqrt{-g}\ drd\theta d\phi \ . \tag{9-14}$$

The use of Eq. (4-8) gives

$$V = e^{3M/2}(dr)(rd\theta)(r\ \sin\theta d\phi). \tag{9-15}$$

But $$e^M = e^G(1+\tau) \tag{9-16}$$

so that $V = \overline{V}(1+\tau)^{3/2}$ (9-17)

where $\overline{V} = e^{3G/2}(dr)(rd\theta)(r \sin\theta d\phi)$.

The change of mass ΔM can occur because of the work done by the pressure against the change of volume and by the flux of mass out of the volume. If this flux is temporarily ignored (i.e. all mass moves along geodesics), then ΔM can be expressed as

$$\Delta M = - \frac{P}{c^2} \Delta V \tag{9-18}$$

where P is the total relativistic pressure. The relation

$$P = \frac{1}{3} c^2 D \tag{9-19}$$

implies that

$$\Delta M = - \frac{1}{3} M \frac{\Delta V}{V} . \tag{9-20}$$

Then Eq. (9-12) becomes, using Eq. (9-20)

$$\Delta D = - \frac{4}{3} D \frac{\Delta V}{V} . \tag{9-21}$$

But Eq. (9-17) with small becomes

$$V = \overline{V} + \frac{3}{2} \overline{V}\tau . \tag{9-22}$$

If ΔV is replaced by $\frac{3}{2} \overline{V}\tau$, V by $\overline{V}$, ΔD by d, and D by ρ in Eq. (9-21), then this equation becomes

$$d = - 2\rho\tau \quad . \tag{9-23}$$

Hence, the correlation function $<\tau d>$ and $<(\partial_0\tau)d>$ become

$$\langle\tau d\rangle = -\,2\rho\langle\tau^2\rangle$$

and

$$\langle(\partial_0\tau)d\rangle = -\,\rho\partial_0\langle\tau^2\rangle \quad . \tag{9-24}$$

In Appendix I, the effect of "mass leakage" from the volume is approximated by means of two variable parameters λ and β. It is shown that a reasonable form for the correlations $\langle\tau d\rangle$ and $\langle(\partial_0\tau)d\rangle$ are

$$\langle\tau d\rangle = -\,2\lambda\rho\langle\tau^2\rangle$$

$$\langle(\partial_0\tau)d\rangle = -\,\beta\rho\partial_0\langle\tau^2\rangle \ . \tag{9-25}$$

The coefficients λ and β may be thought of as representing a zeroth approximation of those functions of x^0, which correct the zero leakage correlation Eqs. (9-24). These coefficients, then, become open parameters which characterize a given model. Certainly, if mass leakage were not important, then λ and β would both be one.

The substitution of Eqs. (9-25) into the dynamic Eqs. (9-8) and (9-9) and into the growth Eqs. (9-4) and (9-5) gives

$$\kappa\rho - \frac{5}{2}\lambda\kappa\rho\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle + \frac{3}{8}\frac{R'}{R}\partial_0\langle\tau^2\rangle$$

$$- \frac{3}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{9-26}$$

$$-\frac{\kappa\rho}{3} + \frac{1}{2}\lambda\kappa\rho\langle\tau^2\rangle - \frac{1}{4}\langle(\partial_0\tau)^2\rangle + \frac{1}{8}\frac{R'}{R}\partial_0\langle\tau^2\rangle$$

$$- \frac{1}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \tag{9-27}$$

$$\partial_0^2\langle\tau^2\rangle + \frac{2R'}{R}\partial_0\langle\tau^2\rangle + \frac{8}{3}\lambda\kappa\rho\langle\tau^2\rangle = 2\langle(\partial_0\tau)^2\rangle \quad (9\text{-}28)$$

$$\partial_0\langle(\partial_0\tau)^2\rangle + \frac{4R'}{R}\langle(\partial_0\tau)^2\rangle = -\frac{4}{3}\beta\kappa\rho\partial_0\langle\tau^2\rangle. \quad (9\text{-}29)$$

3. Solution of the Moment Equations which Govern Growth of the Fluctuations

The four Eqs. (9-26) through (9-29) now contain the four variables ρ, R, $\langle\tau^2\rangle$, and $\langle(\partial_0\tau)^2\rangle$ and the system is now closed. However, the solution of these four equations is not easy because they are nonlinear and nonlinear problems do not in general yield easily to analysis. But it is possible to take advantage of the requirement that the moments which appear in Eqs. (9-26) and (9-27) be small compared to $\kappa\rho$. That is, the additional stress terms cause a small perturbation of the usual, fluctuation free cosmological solutions for ρ and R which are well known. These solutions, derived in Appendix E, are

$$\kappa\rho = -\frac{3a_r}{R^4} \quad (9\text{-}30)$$

and

$$R'^2 = \frac{a_r}{R^2} - k \quad (9\text{-}31)$$

where a_r is a constant of integration and k is the curvature characteristic and may take one of three values -1, 0, +1 (see Ch. IV).

Nonlinear problems can in many cases be simplified by the replacement of a variable with a very close approximation to that variable [30]. If such a replacement is

made for R' and ρ in Eqs. (9-28) and (9-29) with the fluctuation free solutions Eqs. (9-30) and (9-31) and if all derivatives with respect to x^0 are transformed into derivatives with respect to R by the chain rule for differentiation, these two equations become soluble between themselves.

However, as can be seen from Eq. (9-31), the effect of k upon R'^2 is not significant unless R is large enough so that a_r/R^2 is of the same order of magnitude as k. It is well understood that that value of R for which the effect of k is important to the dynamics is the current value of R. For this reason, much research in observational cosmology is devoted to the determination of k on the basis of data on current expansion characteristics [25]. But the applicability of the radiation model is only valid up to the time when the radiation density is equal to the matter density for which the associated R-value is much less than the current value of R by several orders of magnitude [8]. Thus, if the R-domain of the solutions $<\tau^2>(R)$ and $<(\partial_0\tau)^2>(R)$ is restricted to values of R sufficiently less than the current value of R so that $a_r/R^2 \gg |k|$, then the following substitutions for R'^2 and ρ in Eqs. (9-28) and (9-29) are valid.

$$\kappa\rho = -\frac{3a_r}{R^4} \qquad (9\text{-}32)$$

and

$$R'^2 = \frac{a_r}{R^2} \quad . \qquad (9\text{-}33)$$

It should be noted that the neglect of k in cosmological problems is not new. k is neglected, for example, by Hoyle and others in their expression for R'^2 for exactly the same reason that it is neglected here [10].

The growth equations become, using the above approximations

$$R^2\partial_R^2\langle\tau^2\rangle + R\partial_R\langle\tau^2\rangle - 8\lambda\langle\tau^2\rangle = \frac{2R^4}{a_r}\langle(\partial_0\tau)^2\rangle \tag{9-34}$$

$$\partial_R\left[\frac{\langle(\partial_0\tau)^2\rangle}{a_r}\right] + \frac{4}{R}\left[\frac{\langle(\partial_0\tau)^2\rangle}{a_r}\right] = \frac{4\beta}{R^4}\partial_R\langle\tau^2\rangle. \tag{9-35}$$

These equations are solved in Appendix J for $\langle\tau^2\rangle$ and $\langle(\partial_0\tau)^2\rangle$. The solutions are

$$\langle\tau^2\rangle = F_{1/3} + \frac{I_{1/3}}{4\beta}R^{i+4} + \frac{D_{1/3}}{4\beta}R^{j+4} \tag{9-36}$$

$$\frac{\langle(\partial_0\tau)^2\rangle}{a_r} = -4\lambda F_{1/3}R^{-4} + I_{1/3}R^i + D_{1/3}R^j \tag{9-37}$$

where $i = -4 + 2\sqrt{2(\lambda+\beta)}$

$j = -4 - 2\sqrt{2(\lambda+\beta)}$

and $I_{1/3}$, $D_{1/3}$, $F_{1/3}$ are arbitrary constants of integration.

4. Special Submodels

a. Discussion

In general, a given combination of the parameters $I_{1/3}$, $D_{1/3}$, and $F_{1/3}$ specify a nonuniform model which is difficult to treat analytically. Yet, there are special values of these parameters which designate models that are

relatively easy to treat and which are interesting. These are the three models for which only one of the three arbitrary parameters are non zero. That is,

$$(1)\ \langle\tau^2\rangle = F_{1/3}\ ;\ \frac{\langle(\partial_0\tau)^2\rangle}{a_r} = -\ 4\lambda F_{1/3}R^{-4} \tag{9-38}$$

$$(2)\ \langle\tau^2\rangle = \frac{I_{1/3}}{4\beta}R^{i+4}\ ;\ \frac{\langle(\partial_0\tau)^2\rangle}{a_r} = I_{1/3}R^{i} \tag{9-39}$$

$$(3)\ \langle\tau^2\rangle = \frac{D_{1/3}}{4\beta}R^{j+4}\ ;\ \frac{\langle(\partial_0\tau)^2\rangle}{a_r} = D_{1/3}R^{j}\ . \tag{9-40}$$

Actually, not much generality is lost by considering the above three models separately. The $I_{1/3}$, $D_{1/3}$, and $F_{1/3}$ terms of the general Eqs. (9-36) and (9-37) are functions of R to a power. In general, however, these powers are quite different from one another (i.e. try λ, β equal to one which are the zero leakage values for λ and β). Because R changes by many orders of magnitudes during expansions and contractions, only one of the terms would dominate anyway (except for those "particular" values of R for which two or even three of the terms would be of the same order). Hence, most of the models with a given combination of $I_{1/3}$. $D_{1/3}$, and $F_{1/3}$ parameters would degenerate into one of the above three models for almost all R-values.

However, there is a difficulty with the $F_{1/3}$-model. Equation (9-25) says

$$\langle\tau d\rangle = -\ 2\lambda\rho\langle\tau^2\rangle\ .$$

λ must therefore be a parameter which measures the response of the radiation fluid to a given geometric perturbation. It seems unreasonable to suppose that the fluctuation d of the total mass density is positive for positive τ. That is, it seems that only models that are physically reasonable are the ones for which λ is positive. But, from Eq. (9-38), only the negative λ values permit the obviously positive functions $<(\partial_0\tau)^2>$ and $<\tau^2>$ to be positive (a_r is positive by virtue of Eq. (9-32)). Hence, the $F_{1/3}$-model does not seem to have much physical meaning and is therefore not considered in this work. (If some positive λ values would be acceptable then there would be grounds for exploring the content of the $F_{1/3}$-model.)

The two remaining models, namely the $I_{1/3}$ and $D_{1/3}$-models, are very different from one another. The $I_{1/3}$-model represents a model which becomes more nonuniform with increasing R while the $D_{1/3}$-model depicts a model for which the nonuniformity decreases with increasing R. Because there is no intrinsic reason why positive λ values are not acceptable, both of these models can be considered. ($I_{1/3}$ is used to denote the arbitrary constant of an <u>increasing</u> type perturbation field of the metric for a radiation, α equal 1/3 universe and $D_{1/3}$ is used to denote the <u>decreasing</u> type. The same nomenclature will be used to specify the increasing and decreasing type perturbations of the dust, α equal zero universe but with a zero subscript.)

It is of interest to see how the fluctuation stresses which correspond to both the $I_{1/3}$ and $D_{1/3}$-type perturbations act upon the average total mass density ρ and the average scale length R. This can be seen if the stress terms which appear with the average density ρ in Eqs. (9-26) and (9-27) are expressed in a form which corresponds to the functional form of the $I_{1/3}$ and $D_{1/3}$-type fluctuations given by Eqs. (9-39) and (9-40). Equations (9-26) and (9-27) become (with $\frac{k}{R^2} <\tau^2>$ neglected to $\kappa\rho<\tau^2>$ for reasons already stated)

($I_{1/3}$-fluctuations)

$$\kappa\rho + C(I_{1/3})R^{i} \frac{[60\lambda + 24\beta + 6\sqrt{2(\lambda+\beta)}]}{72}$$

$$= - \frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{9-41}$$

$$- \frac{\kappa\rho}{3} - C(I_{1/3})R^{i} \frac{[12\lambda + 8\beta - 2\sqrt{2(\lambda+\beta)}]}{72}$$

$$= - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \tag{9-42}$$

($D_{1/3}$-fluctuations)

$$\kappa\rho + C(D_{1/3})R^{j} \frac{[60\lambda + 24\beta - 6\sqrt{2(\lambda+\beta)}]}{72}$$

$$= - \frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{9-43}$$

$$- \frac{\kappa}{3}\rho - C(D_{1/3})R^{j} \frac{[12\lambda + 8\beta + 2\sqrt{2(\lambda+\beta)}]}{72}$$

$$= - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \tag{9-44}$$

where $C(I_{1/3}) = \frac{9\, I_{1/3}\, a_r}{4\beta}$ and $C(D_{1/3}) = \frac{9\, D_{1/3}\, a_r}{4\beta}$

$$i = -4 + 2\sqrt{2(\lambda+\beta)} \text{ and } j = -4 - 2\sqrt{2(\lambda+\beta)} \ . \quad (9\text{-}45)$$

It should be pointed out that of the three parameters $C(A)$, λ, and β of Eq. (9-45) (where A represents $I_{1/3}$ or $D_{1/3}$), only C_A lends itself to physical measurement. It must not be forgotten that C_A is proportional to a parameter which is the result of an integration and therefore is an arbitrary constant which must be fixed by some observation. But λ and β are not the result of any integration. They are variable parameters which, in a sense, measure the mathematical uncertainty between two pairs of correlation functions (see Eqs. (9-25)). If it were known how to exactly determine these relations, then β and λ would be known numbers and hence be of nonvariable status. As it is, they have to exist as variable parameters. (In the dust model considered later, the dynamics of the matter are determined exactly. Hence, the correlation functions can be found precisely and λ and β do in fact, for that case, become known numbers.)

The solution of the dynamic Eqs. (9-41) to (9-44) are given in the next section.

b. Classification

The question naturally arises: What is the effect of a given combination of fluctuation parameters $C(A)$, λ, and β upon ρ and R? It has already been pointed out in Section 3 of this chapter that the effect of the

fluctuation stresses upon the average density ρ and scale length R is small. Thus, the actual solutions are small perturbations of the fluctuation free solutions and it therefore seems appropriate to compare the solutions of Eqs. (9-41) through (9-44) for ρ and R with these known fluctuation free solutions.

This can be done in the following manner: There are only four distinct ways that the fluctuation stresses can perturb the solutions for ρ and R from their fluctuation free solutions. That is, at a given time x^0, the density ρ and the scale length R can each be perturbed above or below their fluctuation free values. Let each of these four type of perturbations be designated by the letters K, L, M, and N in the manner shown in Fig. 1 for the increasing $I_{1/3}$-fluctuations and Fig. 2 for the decreasing $D_{1/3}$-fluctuations. In both cases, a given letter indicates whether the ρ and R are greater or smaller than the fluctuation free solutions for ρ and R respectively. For example, a model for which $\rho(x^0)$ and $R(x^0)$ are both greater than their respective unperturbed values $\rho^*(x^0)$ and R^* would be classified as type M.

The manner in which the solutions ρ and R are perturbed from their fluctuation free values which are specified by the above simple classification scheme is determined solely by the two parameters λ and β. The magnitude by which a perturbation of a given type occurs is determined by the parameter C(A). Thus, λ and β determine

the qualitative behavior of the model while C(A) determines its quantitative behavior.

In order to determine the classification of a model which is characterized by the three parameters C(A), λ, and β where A refers to either the increasing $I_{1/3}$-type perturbation or the decreasing $D_{1/3}$-type, the solutions for ρ and R are needed. In Appendix L the solutions for $\kappa\rho$ and R'^2 are given in terms of some special parameters f_A, b_A, and m_A which are functions of only the three parameters C(A), λ, and β. These solutions are

$$\kappa\rho = - 3a_r R^{-4} - f_A R^{m_A} \tag{9-46}$$

$$R'^2 = a_r R^{-2} + b_A R^{m_A+2} \quad . \tag{9-47}$$

The unperturbed solutions ρ^* and R^* are those for which f_A and b_A are both equal to zero. It is easily seen that the direction of displacement of ρ from ρ^* is determined solely by the sign of f_A. The sign of the parameter b_A, on the other hand, tells whether R'^2 is larger or smaller than its unperturbed value $(R'^2)^*$ for all values of R. If, in the case of the $I_{1/3}$-type model, a forward time integration of Eq. (9-47) for various values of $b_{I_{1/3}}$ is made from R equal zero, a family of solutions $R(x^0)$ can be generated. The displacement of any one of these solutions from R^* depends on the ratio $R'^2/(R'^2)^*$. For example, if this ratio is greater than one for all R-values, then R is necessarily perturbed above R^* because all solutions of

the family are "pinned" to the same initial value. Because the size of this ratio compared to one is determined by $b_{I_{1/3}}$, it follows that $b_{I_{1/3}}$ also determines the direction of the displacement of R from R*. It can be shown that the sign of $b_{D_{1/3}}$ also determines the displacement of the decreasing $D_{1/3}$ solution of R from R* if the integration is begun at a sufficiently large R and performed backwards in time. In Appendix L, the signs of f_A and b_A (and hence the classification letters) are determined for all values of λ and β for both the $D_{1/3}$ and $I_{1/3}$ type fluctuations. The results are graphically displayed in Figs. 3 and 4.

It might be asked how a given combination of λ and β affect the exponent of R in the fluctuation stress term. The dependence of this exponent is presented in Fig. 5 as a plot of the lines of constant j and i in λ-β space.

This concludes the basic discussion and calculations for the radiation model. In the next section, a similar treatment is made for the dust model. The discussion and comparison of these two models are given in Chapter X.

B. DUST FILLED UNIVERSE MODEL

Some of the calculations and discussions in this section are similar to ones of the previous section which deals with the radiation filled universe for which α is equal to one third. Here the model for which α is equal to zero is treated. It is assumed that the reader is already familiar

with the previous section so that otherwise redundant explanations may be omitted.

For the pressure free universe, Eqs. (7-4), (7-5), (7-6), (7-7), and (7-8) become

$$\kappa\rho - \frac{3}{2}\frac{R'}{R}\partial_0\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle$$
$$- \frac{5}{4}e^{-G}\left\{\langle(\partial_r\tau)^2\rangle + \frac{1}{r^2}\langle(\partial_\theta\tau)^2\rangle + \frac{1}{r^2\sin^2\theta}\langle(\partial_\phi\tau)^2\rangle\right\}$$
$$+ \frac{3k\langle\tau^2\rangle}{R^2} = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{9-48}$$

$$-\frac{3}{2}\frac{R'}{R}\partial_0\langle\tau^2\rangle - \frac{1}{2}\partial_0^2\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle$$
$$- \frac{5}{12}e^{-G}\left\{\langle(\partial_r\tau)^2\rangle + \frac{1}{r^2}\langle(\partial_\theta\tau)^2\rangle + \frac{1}{r^2\sin^2\theta}\langle(\partial_\phi\tau)^2\rangle\right\}$$
$$+ \frac{k}{R^2}\langle\tau^2\rangle = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \tag{9-49}$$

$$-\frac{5}{4}e^{-G}\left\{\langle(\partial_r\tau)^2\rangle + \frac{1}{r^2}\langle(\partial_\theta\tau)^2\rangle + \frac{1}{r^2\sin^2\theta}\langle(\partial_\phi\tau)^2\rangle\right.$$
$$= -\frac{5}{2}\kappa\rho - \frac{5}{2}\kappa\langle\tau d\rangle - \frac{15k}{R^2} - \frac{15R'^2}{R^2} - \frac{15R''}{R} \tag{9-50}$$

$$\kappa\langle\tau d\rangle - \frac{3}{2}\partial_0^2\langle\tau^2\rangle - \frac{3R'}{R}\partial_0\langle\tau^2\rangle = -3\langle(\partial_0\tau)^2\rangle \tag{9-51}$$

$$\partial_0\langle(\partial_0\tau)^2\rangle + \frac{4R'}{R}\langle(\partial_0\tau)^2\rangle = \frac{2}{3}\kappa\langle(\partial_0\tau)d\rangle \tag{9-52}$$

where

$$\rho \equiv \langle D\rangle$$

and

$$d \equiv D - \rho \quad . \tag{9-53}$$

It should be noted that for α equal to zero, the pressure is identically zero. Hence, there are no pressure gradients available to accelerate the matter. This implies that matter, when left alone, will follow geodesic lines. If these geodesics are taken to be geodesics of zero velocity then the matter comoves with the coordinate system (r,θ,ϕ) and the above equations follow. From Eqs.(9-53) it is seen that ρ is the average of the proper density $\bar{D}$ because the relative velocity u is zero in which case D is equal to $\bar{D}$ (see Eq. (5-20)).

1. Elimination of the Spatial Derivatives

In Eqs. (9-48) and (9-49), the terms which contain spatial partial derivatives may be eliminated explicitly in a manner analogous to the way they are removed from Eqs. (9-1) and (9-2). This computation is performed in Appendix G. The resulting two equations are

$$\kappa\rho + \frac{3}{4} \langle(\partial_0\tau)^2\rangle + \frac{3}{8}\frac{R'}{R}\partial_0\langle\tau^2\rangle + \frac{5}{4}\kappa\langle\tau d\rangle - \frac{3}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{9-54}$$

$$-\frac{1}{4}\langle(\partial_0\tau)^2\rangle + \frac{1}{8}\frac{R'}{R}\partial_0\langle\tau^2\rangle + \frac{1}{12}\kappa\langle\tau d\rangle - \frac{1}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} . \tag{9-55}$$

2. Inclusion of the Dynamics for the Matter

In order to solve the system of equations, it is necessary to express $\langle\tau d\rangle$ in terms of existing correlation

functions. This is also done in the previous section but is complicated by the existence of nonzero velocities. For the pressure free model this calculation is much easier.

Consider Eq. (9-12). Because the velocities and the pressure are zero, there is no mechanism to change the mass content of an expanding volume. Hence, ΔM must be zero and Eq. (9-12) becomes

$$\Delta D = - \frac{M\Delta V}{V^2} \quad . \tag{9-56}$$

With Eq. (9-22) for small τ, Eq. (9-56) becomes (also with the usual definitions for the increments as in the previous section)

$$d = - \frac{3}{2} \rho\tau \quad . \tag{9-57}$$

Hence, $<\tau d>$ and $<(\partial_0 \tau)d>$ must be

$$<\tau d> = - \frac{3}{2} \rho<\tau^2> \tag{9-58}$$

and

$$<(\partial_0 \tau)d> = \frac{-3}{4} \rho\partial_0<\tau^2> \quad . \tag{9-59}$$

Equation (9-57) is also derived in Appendix I using the conservation equations

$$T^{\mu\nu}{}_{;\,\nu} = 0 \ .$$

It should be noted that there is no need for any arbitrary parameters λ and β. This is because the above dynamical equations can be determined exactly.

With Eqs. (9-58) and (9-59), the dynamic Eqs. (9-54) and (9-55) and the growth Eqs. (9-51) and (9-52) become,

$$\kappa\rho + \frac{3}{4}<(\partial_0\tau)^2> + \frac{3}{8}\frac{R'}{R}\partial_0<\tau^2> - \frac{15}{8}\kappa\rho<\tau^2>$$

$$- \frac{3}{4}\frac{k}{R^2}<\tau^2> = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \qquad (9\text{-}60)$$

$$-\frac{1}{4}<(\partial_0\tau)^2> + \frac{1}{8}\frac{R'}{R}\partial_0<\tau^2> - \frac{1}{8}\kappa\rho<\tau^2>$$

$$-\frac{1}{4}\frac{k}{R^2}<\tau^2> = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \qquad (9\text{-}61)$$

$$\partial_{RR}<\tau^2> + \left(\frac{R''}{R'^2} + \frac{2}{R}\right)\partial_R<\tau^2> + \frac{\kappa\rho}{R'^2}<\tau^2>$$

$$= \frac{2}{R'^2}<(\partial_0\tau)^2> \qquad (9\text{-}62)$$

$$\partial_R<(\partial_0\tau)^2> + \frac{4}{R}<(\partial_0\tau)^2> = -\frac{\kappa}{2}\rho\partial_R<\tau^2> \ . \qquad (9\text{-}63)$$

3. <u>Solution of the Moment Equations which Govern the Growth of the Fluctuations</u>

The growth equations are solved in Appendix K. The interested reader might note the differences between the growth equations for the pressure free model and the radiative pressure model. The most significant difference is that the curvature constant k is in general not negligible in the region where the matter density exceeds the radiation density. Therefore, in order to avoid difficult integration problems, the growth equations are solved only for the k equal zero universe. The advantage for doing

this is that the results can be expressed in terms of simple functions. (It seems that such problems presented by a nonvanishing k are inherent to nonuniform models. Sachs and Wolfe [24], for example, are also led to consider only Euclidean models for the same reason.) Nevertheless, the results for k equal to zero should have some relevance to k equal to plus and minus one universes. This is because the dynamics of the universe are now only slightly modified by curvature effects [25] and the solutions to the growth equations ought to therefore be approximately correct for such models also.

With k equal to zero, the fluctuation free solutions for $\kappa\rho$ and R'^2 given in Appendix E are

$$\kappa\rho = -\frac{3a_d}{R^3}$$

and

$$R'^2 = \frac{a_d}{R} \tag{9-64}$$

where a_d is an integration constant different from a_r which appears in the analogous equations for the radiation filled universe.

Then the growth equations become

$$R^2\partial_{RR}\langle\tau^2\rangle + \frac{3}{2}R\partial_R\langle\tau^2\rangle - 3\langle\tau^2\rangle$$

$$= \left(\frac{2}{a_d}\right) R^3\langle(\partial_0\tau)^2\rangle \tag{9-65}$$

and

$$\partial_R\langle(\partial_0\tau)^2\rangle + \frac{4}{R}\langle(\partial_0\tau)^2\rangle = \left(\frac{3a_d}{2}\right)\frac{1}{R^3}\partial_R\langle\tau^2\rangle. \tag{9-66}$$

The solutions to these equations are given in Appendix K.

$$\frac{\langle(\partial_0\tau)^2\rangle}{a_d} = I_0 R^{-1} + F_0 R^{-7/2} + D_0 R^{-6} \tag{9-67}$$

$$\langle\tau^2\rangle = I_0 R^2 - \frac{2}{3} F_0 R^{-1/2} + \frac{4}{9} D_0 R^{-3} . \tag{9-68}$$

4. Special Submodels

Again, as in the radiation model, only the following three models are considered.

$$(1)\ \langle\tau^2\rangle = I_0 R^2 \ ; \ \frac{\langle(\partial_0\tau)^2\rangle}{a_d} = I_0 R^{-1} \tag{9-69}$$

$$(2)\ \langle\tau^2\rangle = -\frac{2}{3} F_0 R^{-1/2} \ ; \ \frac{\langle(\partial_0\tau)^2\rangle}{a_d} = F_0 R^{-7/2} \tag{9-70}$$

$$(3)\ \langle\tau^2\rangle = \frac{4}{9} D_0 R^{-3} \ ; \ \frac{\langle(\partial_0\tau)^2\rangle}{a_d} = D_0 R^{-6} . \tag{9-71}$$

The same difficulty occurs with the F_0-model as with the $F_{1/3}$-model of the previous section. That is, it is impossible to find a nonzero F_0 with a sign that is compatible with both of the Eqs. (9-70) (a_d is always greater than zero). Hence, only the I_0 and D_0 models are considered. It is to be noted that the I_0-model is a model for which the nonuniformity increases with increasing R and the D_0-model is a model for which it decreases with increasing R.

If Eqs. (9-69) and (9-71) are substituted into Eqs. (9-60) and (9-61) the following equations are obtained.

(I_0 - fluctuations)

$$\kappa\rho + 57\ B(I_0)R^{-1} = -\frac{3R'^2}{R^2}$$

$$3\ B(I_0)R^{-1} = -\frac{2R''}{R} - \frac{R'^2}{R^2} \tag{9-72}$$

(D_0 - fluctuations)

$$\kappa\rho + 22\ B(D_0)R^{-6} = -\frac{3R'^2}{R^2}$$

$$-2\ B(D_0)R^{-6} = -\frac{2R''}{R} - \frac{R'^2}{R^2} \tag{9-73}$$

where

$$B(I_0) = \frac{I_0 a_d}{8}$$

and

$$B(D_0) = \frac{D_0 a_d}{8} \quad . \tag{9-74}$$

The solutions of Eqs. (9-72) and (9-73) turn out to be

(I_0 - fluctuations)

$$\kappa\rho = -\frac{3a_d}{R^3} - \frac{105}{2}\frac{B(I_0)}{R} \tag{9-75}$$

$$R'^2 = \frac{a_d}{R} - \frac{3}{2}B(I_0)R \tag{9-76}$$

(D_0 - fluctuations)

$$\kappa\rho = -\frac{3a_d}{R^3} - \frac{20\ B(D_0)}{R^6} \tag{9-77}$$

$$R'^2 = \frac{a_d}{R} - \frac{2\ B(D_0)}{3}\frac{1}{R^4} \quad . \tag{9-78}$$

If the same classification scheme, which is depicted in Figs. 1 and 2, is used to classify these solutions that is used to classify the solutions for the radiation model, then it is easily found that the I_0 - fluctuations are of type K and the D_0 - fluctuations are of type M. The discussion of this model and the radiation model is given in the next chapter.

X. SUMMARY AND CONCLUSIONS

A. VALIDATION OF PROCEDURE

Before discussing the results of this study regarding the development of a cosmological model to demonstrate the applicability of a moment hierarchy scheme to perturbation problems of general relativity, it would be well to show that the solutions obtained are compatible with ones obtained by a completely different procedure.

Well known in the literature is the work by Sachs and Wolfe [24] concerning the anisotropy of the microwave 3°K background radiation due to perturbations of geometry. In this study a scheme is presented for analyzing small, linear perturbations away from an unperturbed, k equal zero Robertson-Walker metric. The method of solving for the perturbations is to first take the spatial Fourier transform of the perturbations, solve them in $\vec{k}$ space, and then transform them back into position space. They do this for both the dust and radiation models.

For the dust model (pressure free) they find two kinds of terms. In both types, $\delta\rho$ decreases although $\delta\rho/\rho$ increases for one and decreases for the other (in their notation, ρ is the mass density and $\delta\rho$ is its small perturbation). Furthermore, the characteristic time for their relative increasing solution is of the same order as the characteristic time for the background metric. They report their

solutions for $\delta\rho$ to be of the form

$$\delta\rho = DR^{-9/2} + IR^{-2} \tag{10-1}$$

where D and I are functions only of the spatial coordinates. Because ρ goes as I/R^3,

$$\frac{\delta\rho}{\rho} = DR^{-3/2} + IR \tag{10-2}$$

where the constant of proportionality is absorbed into D and I.

As shown by Eq. (9-57), the ratio of $\delta\rho/\rho$ must be proportional to τ. Hence, according to Eq. (10-2), τ must also have the form

$$\tau = DR^{-3/2} + IR. \tag{10-3}$$

Thus, τ^2 must be given by

$$\tau^2 = D^2R^{-3} + (DI)R^{-1/2} + I^2R^2 \ . \tag{10-4}$$

But Eq. (9-68) says that

$$\langle\tau^2\rangle = \frac{4}{9} D_0 \ R^{-3} + \frac{-2F_0}{3} R^{-1/2} + I_0R^2 \ . \tag{10-5}$$

It is seen that the exponents of each term agree with those of Sachs and Wolfe. This agreement is noteworthy because Eqs. (10-4) and (10-5) are produced by two very different procedures.

The determination of the characteristic time T_I for the I_0 (increasing) perturbation term τ can be obtained from the ratio of $\langle\tau^2\rangle$ to $\langle(\partial_0\tau)^2\rangle$. That is,

$$T_I^2 = \frac{1}{c^2} \frac{< \tau^2 >}{<(\partial_0 \tau)^2>} \tag{10-6}$$

where c is the speed of light. Using Eq. (9-69)

$$T_I^2 = \frac{R^3}{c^2 a_d} \; . \tag{10-7}$$

The characteristic time T_B for the background geometry can be found from the ratio of R^2 to $(R')^2$. That is,

$$T_B^2 = \frac{1}{c^2} \frac{R^2}{R'^2} \; . \tag{10-8}$$

With Eq. (9-64), T_B^2 becomes

$$T_B^2 = \frac{R^3}{c^2 a_d} \; . \tag{10-9}$$

Indeed, the ratio $\delta\rho/\rho$ which is proportional to τ is predicted by the model to take place on the same time scale as the background because the ratio of T_I to T_B is one. This constitutes a second point of agreement with Sachs and Wolfe.

The above comparisons are made only for the dust models. But a comparison between radiation models can be made in a similar manner. Sachs and Wolfe indicate that they neglect transport processes in their derivation. Hence, the corresponding model of this work is the "zero leakage" model for which λ and β are both equal to one.

For their radiation model, Sachs and Wolfe report that

$$\delta\rho = DR^{-6} \tag{10-10}$$

so that

$$\left(\frac{\delta\rho}{\rho}\right)^2 = D^2R^{-4} \tag{10-11}$$

where D is a function only of the spatial coordinates.

The general solution for $<\tau^2>$ given by Eq. (9-36) with λ and β equal to one is

$$<\tau^2> = \frac{I_{1/3}}{4} R^4 + \frac{D_{1/3}}{4} R^{-4} \ . \tag{10-12}$$

The $F_{1/3}$ solution is omitted from the general solution of Eq. (10-12) because, as stated in the text, certain sign problems are associated with it.

But Eq. (9-24) indicates that

$$\frac{<\tau d>}{\rho} = - 2<\tau^2> \tag{10-14}$$

This indicates that

$$\left(\frac{d}{\rho}\right)^2 \alpha \ \tau^2 \tag{10-15}$$

That is,

$$\left(\frac{d}{\rho}\right)^2 \alpha \left(\frac{I_{1/3}}{4}\right) R^4 + \left(\frac{D_{1/3}}{4}\right) R^{-4} \ . \tag{10-16}$$

The comparison between Eqs. (10-11) and (10-16) indicates that although there is disagreement concerning the existence of an R^4 term, there is agreement concerning the R^{-4} term. It is not possible to assess the reason for this discrepency owing to the omission of the solution procedure from Sach's and Wolfe's paper. However, it is significant to this work that with all the possibilities for the values of the

exponents due to the many possibilities for λ and β, there is an agreement concerning the existence of an R^{-4} term.

Thus, there is a major agreement between the functional forms of both the k equal to zero, dust and radiation filled models. Because this agreement is accomplished by two very different perturbation techniques, support is given to the method of this work for treating perturbations by a hierarchy of moment equations.

It should again be mentioned that the agreement of the radiation model is contingent upon the selection of the "zero leakage" values for λ and β because Sachs and Wolfe assume that the velocities are zero. It is reasonable to ask how valid is this assumption. In the derivation of the model, the attempt was made to include the effects of the velocities by the introduction of the parameters λ and β whose magnitudes are estimated in Appendix H to be of order one.

Figures 3 and 4 give the types of solutions that can be expected for all possible combinations of λ and β. In the case of the $I_{1/3}$ - type perturbation, the model classification for λ and β values of order one is clearly type K. But for the $D_{1/3}$ - type, both the K and M - type solutions are possible. In fact, the zero leakage values of λ and β are observed to lie on the boundary between the K and M - type solutions. Thus, if a $D_{1/3}$ - type model is being considered, the neglect of the velocities is not trivial for very different dynamical situations could occur even if the velocities

are extremely small. But for an $I_{1/3}$ - type, the neglect of the velocities does not change the solution type. In this case the approximation of zero velocities seems to be justified. Indeed, Sachs and Wolfe consider only this type.

However, λ and β not only affect the classification but also determine the exponent (which is $j+4$ for $D_{1/3}$ and $i+4$ for $I_{1/3}$) of R in the expression for $<\tau^2>$. If Figure 5 the lines of constant i and j are given in λ-β space. From these curves, it appears that for λ and β of order one,

$$-3 \lesssim j + 4 \lesssim -8$$

$$3 \lesssim i + 4 \lesssim 8 \ . \qquad (10\text{-}17)$$

In the case of λ and β equal to exactly one, $j+4$ is equal to -4 and $i+4$ equal to 4. These estimates are conservative. Perhaps a more reasonable estimate would be to assume that the likelihood of λ and β both being on the large side of their order one magnitude is small. Then the upper limit might be pushed down to perhaps -6 and 6 respectively.

Thus it seems that the zero velocity assumption can reasonably be questioned on the grounds that the exponent of $<\tau^2>$ could be different enough from the zero velocity exponent that a many order of magnitude change of R could produce a value of $<\tau^2>$ that is much larger (or smaller) than the corresponding zero velocity value. This can be seen more quantitatively as follows. Consider, for example, a radiation model with an $I_{1/3}$ - type perturbation. At some value of R equal to $\overline{R}$,

$$\langle\tau^2\rangle_i = \frac{I^i_{1/3}}{4\beta}\,\bar{R}^{i+4} \tag{10-18}$$

where the sub and superscript i refers to the i-value of the exponent. If it is assumed that the ratio $\langle\tau^2\rangle_i/\langle\tau^2\rangle_0$ is one at $\bar{R}$, then this ratio is in general different at other values of R. That is, this ratio is a measure of the effect of a given combination of λ and β upon the nonuniformity in terms of the "zero velocity" nonuniformity. This ratio is easily found to be, for an arbitrary R

$$\frac{\langle\tau^2\rangle_i}{\langle\tau^2\rangle_0} = \left(\frac{R}{\bar{R}}\right)^i \tag{10-19}$$

where $i = -4 + 2\sqrt{2(\lambda+\beta)}$.

Thus, unless λ and β are such that i is near enough to zero so that $\langle\tau^2\rangle$ is of order $\langle\tau^2\rangle_0$, the effects of the expansion through many orders of magnitude upon the nonuniformity compared to the zero velocity nonuniformity is significantly increased as seen from Eq. (10-19). If, however, R changes at most by only a few orders of magnitude in a given problem, the discrepency is not as serious and may be tolerable if i is close enough to 0. Hence, although the zero velocity approximation is extremely convenient mathematically, its validity does not appear to be on solid footing.

B. DESCRIPTION AND COMPARISON OF MODELS

As stated in Section A of this chapter, there are two intrinsically different types of nonuniformities associated with both the dust and radiation filled models. One is an

increasing (I-type) and the other is a decreasing (D-type) nonuniformity. If the ratio of $<\tau^2>$ to $<(\partial_0\tau)^2>$ is computed for all four models (two for each α value) in the same way as done previously so that a characteristic time might be estimated for the relative perturbations, then it is found that, for all models, this characteristic time is of the same order as the characteristic time for the average background.

Because of the existence of the two D and I-type perturbations, it is natural to ask which kind characterizes our universe. This question can only be answered through observation. Although such observations are difficult to perform, there seems to be some suggestion that the I-type of perturbation is the one particular to our own universe. One reason why this might be true is that the average background metric might be expanding more rapidly than the supercluster to which our galaxy belongs. A second reason is that, in the case of D-type perturbations, galaxies might be discarded from their otherwise clustered configurations [24].

Granting the I-type dust model, it is possible to estimate the parameter I_0 which governs the amplitude of the metric perturbation $<\tau^2>$. This can be accomplished by using the data reported by Partridge and Wilkinson [19] on the anisotropy of the so-called cosmic blackbody radiation. They report that the ratio $\delta T/T$ is of the order 10^{-3} where T is the temperature of the radiation (3°K) and δT is the fluctuation of T with angle of observation. Sachs and Wolfe [24]

give a very simple expression that relates $\delta T/T$ to $\delta\rho_0/\rho_0$ where ρ_0 is the current mass density.

$$\frac{\delta T}{T} \approx \frac{1}{2} \frac{\delta\rho_0}{\rho_0} (H_R L)^2 \tag{10-20}$$

where $H_R = \frac{R_0{}'}{R_0}$ (' denotes derivative wrt. x^0), R_0 is the current value of R, and L is the characteristic length of the perturbations. By Eq. (9-57), Eq. (10-20) can be written as

$$\tau^2 = \frac{16}{9} \frac{1}{(H_R L)^4} \left(\frac{\delta T}{T}\right)^2 \quad . \tag{10-21}$$

The constant I_0 can be estimated by means of Eqs. (9-69) and (10-21). That is,

$$I_0 \approx \frac{\varepsilon^4}{R_0{}^2} \left(\frac{\delta T}{T}\right)^2 \left(\frac{16}{9}\right) \frac{1}{(H_R R_0)^4} \tag{10-22}$$

where

$$\varepsilon \equiv \frac{R_0}{L} \quad . \tag{10-23}$$

If $H_R R_0$ is taken to be one and if the factor 16/9 is neglected since it is of order one, Eq. (10-22) becomes simply,

$$I_0 \approx \frac{\varepsilon^4}{R_0{}^2} \left(\frac{\delta T}{T}\right)^2 . \tag{10-24}$$

If the fluctuations are assumed to be the size of about 1000 MPC [24, 31] (one MPC is equal to 3.26 light years) so that ε is of order ten ($R_0 \approx 10^{28}$ cm), then for Partridge and Wilkinson's estimate of $\delta T/T$, Eq. (10-24) gives for I_0

$$I_0 \approx 10^{-58}\text{cm}^{-2} \quad . \tag{10-25}$$

Using this value for I_0, it is interesting to determine the effect a term of this strength would have upon the average dynamics if it is assumed that the long wavelengths are important to the expansion characteristics [35]. It is well known that the k = 0, unperturbed uniform universe model expands forever [2]. (This can be seen easily by integrating Eq. (9-64) to find $R \propto t^{2/3}$.) But owing to the additional source term, the dynamics of the model is perturbed. As pointed out earlier, the solution for the k = 0, I_0 model is type K. Because this means that R is perturbed below the uniform solution, (see Fig. 1), the question naturally arises whether or not there exists a maximum value for R. To answer this, R'^2 must be set to zero in Eq. (9-76) and a positive value of R greater than R_0 must be determined. If this is done, such a value R_{max} is found such that

$$R_{max} = \left(\frac{16}{3I_0}\right)^{1/2} \quad . \tag{10-26}$$

Using Eq. (10-25),

$$R_{max} \approx 10^{29}\text{ cm} \quad . \tag{10-27}$$

That is, if the observed anisotropy of the blackbody radiation is due to perturbations of space as large as supercluster distances and are of the I_0 - type, then the "effective energy" of these longwave fluctuations is

sufficient to prevent an unlimited expansion of a $k = 0$, dust filled universe which would otherwise expand forever.

It should be pointed out that this model is not valid for all values of R. For small R, the radiation density, which goes as R^{-4}, overwhelms the mass density, which goes as R^{-3}. At this point, it is no longer valid to speak of a dust model and it is proper to use the radiation model for smaller R-values [15]. On the other hand, for large R, $\langle\tau^2\rangle$ approaches one. As it does so, the assumption of small amplitude fluctuations which was used to truncate the moment hierarchy, becomes invalid. The point at which this happens can be easily determined. With $\langle\tau^2\rangle$ equal to one, Eq. (9-69) gives

$$R_c = \frac{1}{(I_0)^{1/2}} \tag{10-28}$$

where R_c is that value of R for which $\langle\tau^2\rangle$ is equal to one.

If Eq. (10-28) is compared to Eq. (10-26), it is found that $R_c < R_{max}$ which indicates that at the "turning point," the fluctuations are as large or larger than the average metric. However, it should be noted that because the model is not applicable at the turning point itself, it may not therefore be justifiably used to determine R_{max}. This difficulty is not serious though since the ratio of the "extrapolated value" of R_{max} given by Eq. (10-26) to R_c is

$$\frac{R_{max}}{R_c} = \frac{4}{\sqrt{3}} \tag{10-29}$$

which is of order one. Thus the real value of R_{max} should still be of the order of 10^{29} cm as given by Eq. (10-27) since the theory breaks down close to the estimated value of R_{max}. It should finally be noted that if other models are considered which also have "turning points," (either as a minimum or maximum) the weak fluctuation assumption breaks down just before the turning point occurs. Thus, if the physics of an assumed turning is to be accurately discussed, a strong fluctuation model should be used.

APPENDIX A

The Reynold Stresses

The Navier-Stokes equation for an incompressible fluid is [12]

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{V} \qquad \text{(A-1)}$$

where $\vec{V}$ and P are the velocity and pressure fields relative to a fixed coordinate system and ν is the ratio η/ρ where η is the viscosity coefficient and ρ is the mass density. For incompressible flow, ρ is a constant throughout all space.

In order to gain some insight into the perturbation technique that is presented in this paper, the reader should look upon the Navier-Stokes equation as if it were the Einstein Field equations. The left hand side of Eq. (A-1) is analogous to the geometrical side of the field equations and the right hand side is analogous to the mass side (that is, the side which contains the energy momentum tensor). The velocity field $\vec{V}$ plays the same role in the Navier-Stokes equation as does the metric tensor $g_{\mu\nu}$ in the Einstein field equations.

In fluid turbulence theory, the following decompositions are assumed [4]. (In this appendix, the equation after the semicolan refers to the analogous equation in the text.)

$$\vec{V} = \langle\vec{V}\rangle + \vec{v} \qquad \text{(A-2);(4-9)}$$

$$P = \langle P \rangle + p \qquad \text{(A-3);(4-9)}$$

where < > denotes "average."

If Eqs. (A-2) and (A-3) are substituted into Eq. (A-1), the following equation results

$$\frac{\partial \langle \vec{V} \rangle}{\partial t} + \frac{\partial \vec{v}}{\partial t} + \langle \vec{V} \rangle \cdot \nabla \langle \vec{V} \rangle + \vec{v} \cdot \nabla \langle \vec{V} \rangle$$

$$+ \langle \vec{V} \rangle \cdot \nabla \vec{v} + \vec{v} \cdot \nabla \vec{v}$$

$$= - \frac{1}{\rho} \nabla \langle P \rangle - \frac{1}{\rho} \nabla p + \nu \nabla^2 \langle \vec{V} \rangle + \nu \nabla^2 \vec{v}. \qquad \text{(A-4)}$$

In fluid turbulence theory it is assumed [4] that if A is a random field,

$$\left\langle \frac{\partial A}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle A \rangle \qquad \text{(A-5);(B-2)}$$

and

$$\left\langle \frac{\partial A}{\partial x^i} \right\rangle = \frac{\partial}{\partial x^i} \langle A \rangle . \qquad \text{(A-6);(B-2)}$$

Because of Eqs. (A-2) and (A-3),

$$\langle \vec{v} \rangle = 0 \qquad \text{(A-7);(4-14)}$$

and

$$\langle p \rangle = 0 . \qquad \text{(A-8)}$$

Then, Eq. (A-4) becomes

$$\frac{\partial \langle \vec{V} \rangle}{\partial t} + \langle \vec{V} \rangle \cdot \nabla \langle \vec{V} \rangle = - \frac{1}{\rho} \nabla \langle P \rangle$$

$$+ \nu \nabla^2 \langle \vec{V} \rangle - \langle \vec{v} \cdot \nabla \vec{v} \rangle . \qquad \text{(A-9);(6-8),(6-9)}$$

It should be noted that Eq. (A-9) is symbolically identical with Eq. (A-1) with $\vec{V}$ and P replaced with $\langle \vec{V} \rangle$ and $\langle P \rangle$. The only difference, however, is the appearance of the extra

source term from the "velocity side" which results from the nonlinear $\vec{V} \cdot \nabla\vec{V}$ term. It is, in general, nonvanishing.

This extra term is called the Reynold Stress. It can be interpreted as an additional source term (in addition to the pressure and viscous sources) which acts on the average flow field ($\langle\vec{V}\rangle$). Such an interpretation can be made because the left hand side of Eq. (A-9) is symbolically identical with the left hand side of Eq. (A-1).

In order to obtain the functional form of the Reynold Stresses, a hierarchy of equations can be built up from the Navier-Stokes equation of Eq. (A-1) by multiplying it with various orders of $\vec{V}$ and averaging. But the nonlinear term, as already seen, always produces a moment of one order greater than the order of the moments which comes from the other linear terms. Thus, there is always one more unknown than there are necessary equations available to solve the system of equations. This problem is the well known closure problem of turbulence theory which is discussed in the text [3].

APPENDIX B

Definition of the Averaging Operation

There are various definitions of average which are possible. Many theorists who work in the field of fluid turbulence use an integral expression for their definition of average. For example, Curle and Davies [4] use a time integral. This is an acceptable definition for experimental work but as pointed out by Lumley and Panofsky [13], such an integral expression is not always easy to define in a meaningful way. They suggest that an ensemble average is probably the most basic. Thus, the notion of an ensemble average is taken as the definition for the averaging process $< \; >$ for this model.

Suppose $D(x^\mu, \xi)$ is a stochastic process where x^μ are the four space-time coordinates and ξ is the ensemble parameter which designates D as a possible outcome for an experiment (see also Ch. IV). Then $<D>$ is defined as

$$<D> = \lim_{N\to\infty} \frac{D(x^\mu;\ \xi_1) + \ldots + D(x^\mu;\ \xi_N)}{N} \tag{B-1}$$

where $<D>$ is only a function of x^μ and is not dependent on ξ. This is the same definition as that used by Lumley and Panofsky.

Because this definition involves a summation, the following equation is assumed valid.

$$\partial_\mu \langle D \rangle = \langle \partial_\mu D \rangle \quad . \tag{B-2}$$

Suppose it is desired to know the probability of finding $D(x^\mu)$ falling below a certain value δ. This can be determined by counting the proportion of members of the ensemble for which $D(x^\mu) \leq \delta$. From this a function F can be defined as follows

$$F(\delta; x^\mu) = P\{D(x^\mu) \leq \delta\} \tag{B-3}$$

where P stands for "Probability that." The probability density function is defined by

$$f(D; x^\mu) = \frac{\partial F(\delta; x^\mu)}{\partial \delta} \quad . \tag{B-4}$$

Then the average of any function $h[D(x^\mu)]$ can be written as

$$\langle h[D(x^\mu)] \rangle = \int_{-\infty}^{\infty} h(\delta) f(\delta, x^\mu) d\delta \quad . \tag{B-5}$$

APPENDIX C

An Implication of Spatial Stationarity of the Random Field

An implication of statistical spatial stationarity is rather easy to state because of the assumed form of the line element. On a hypersurface of constant x^0, dx^0 is zero so that by Eqs. (4-5) and (4-6)

$$ds^2\Big|_{x^0} = -\frac{R_M^{\,2}(x^0, x^j; \xi)}{r_0^{\,2}\left(1 + \frac{k}{4}\frac{r^2}{r_0^{\,2}}\right)^2}\, d\sigma^2 \ . \tag{C-1}$$

It is pointed out in Chapter IV that R_M has the meaning of "local radius of curvature." It is also a stochastic process which means that at each point x^μ of space time it has a probability density function associated with it (see Appendix B).

In order to bring about a simplification of the proposed model, consider the following assumption. Although the hyperspaces of constant x^0 are clearly nonuniform, and under go various expansions at each point, it is assumed that the probability of finding a given expansion at any point is independent of position on that hyperspace. This requirement is then one of statistical spatial stationarity, a concept which is familiar in fluid turbulence theory. This implies that the probsbility of measuring a given value of $R_M^{\,2}$ between say $\bar{R}_M^{\,2}$ and $\bar{R}_M^{\,2} + d\bar{R}_M^{\,2}$ ought to be independent

of the coordinates on this spatial hypersurface of constant time x^0 because this random function contains all the expansion characteristics. Equivalently, the probability of finding τ between T and T + dT ought to also be independent of the spatial coordinates. Hence, the probability density function $f(T;x^\mu)$ defined by Eq. (B-4) must be such that,

$$f(T;x^0,x^j) = f(T;x^0,x^{j'})$$

for every x^j and $x^{j'}$ where j = 1, 2, 3. This can only mean that [18]

$$f(T;x^\mu) = f(T;x^0) \tag{C-2}$$

That is, f can only be a function of the coordinate x^0 and τ. But Eq. (B-5) says that for any function h of the process $\tau(x^\mu;\xi)$,

$$\langle h(\tau)\rangle = \int_{-\infty}^{\infty} h(T)\, f(T;x^0)dT \ . \tag{C-3}$$

Hence, $\langle h(\tau)\rangle$ can only depend upon x^0 and not upon r, θ, ϕ. It follows that

$$\partial_j \langle h(\tau)\rangle = 0 \tag{C-4}$$

where j = r, θ, ϕ.

APPENDIX D

The Form of the Energy-Momentum Tensor

The general form of the energy-momentum tensor for a perfect fluid is given by [1, 29]

$$T^{\mu\nu} = \bar{D}U^{\mu}U^{\nu} + \frac{\bar{P}}{c^2}(U^{\mu}U^{\nu} - g^{\mu\nu}) \tag{D-1}$$

where $U^{\mu} = dx^{\mu}/ds$ and $\bar{D}$ and $\bar{P}$ are the proper hydrostatic density and pressure. That is, these are the density and pressure which would be measured by an observer moving instantaneously with the fluid. Tolman [29] defines a perfect fluid to be a mechanical medium incapable of exerting transverse stresses in the proper frame. From this definition follows Eq. (D-1).

Tolman [28] shows that the energy-momentum tensor for a disordered electromagnetic field, which is expressed in terms of the electric and magnetic field strengths, can be viewed as a perfect fluid for which

$$\bar{P} = \frac{1}{3} c^2\bar{D}. \tag{D-2}$$

However, if the pressure is due to the random motion of particles of cold dust, then the equation of state, which is well-known to kinetic theory, is given by [1]

$$P = \frac{1}{3}\frac{V^2}{c^2}\rho \tag{D-3}$$

where V^2 is the root-mean-square random velocity of the dust particles. But this velocity is much less than c so the pressure, for all practical purposes, is essentially zero.

The two models, therefore, can be written together in one equation.

$$\bar{P} = \alpha c^2 \bar{D} \tag{D-4}$$

where $\alpha = 1/3$ if universe is radiation dominated

$\alpha = 0$ if universe is matter dominated .

Now, the expression for the mixed energy momentum tensor $T^{\mu}{}_{\nu}$ is needed. This is easily found by means of the metric tensor $g_{\mu\nu}$.

$$T^{\mu}{}_{\nu} = \bar{D} U^{\mu} U_{\nu} + \frac{\bar{P}}{c^2} (U^{\mu} U_{\nu} - \delta^{\mu}{}_{\nu}) \tag{D-5}$$

where

$$U_{\nu} = g_{\alpha\nu} U^{\alpha} \ . \tag{D-6}$$

In order to evaluate Eq. (D-5), U^{μ} and U_{ν} need to be determined. This can be done as follows. Consider a fluid element which moves along the trajectory $x^j(x^0)$. Along such a trajectory,

$$ds^2 = (dx^0)^2 \left[1 - e^M \frac{d\sigma^2}{(dx^0)^2} \right] \ . \tag{D-7}$$

With the definition

$$u = e^{M/2} \left(\frac{d\sigma}{dx^0} \right) \ , \tag{D-8}$$

equation (D-7) becomes

$$\frac{ds}{dx^0} = \pm (1 - u^2)^{1/2} \quad . \tag{D-9}$$

Notice that if the motion of the fluid element is along a null geodesic, u^2 must be equal to one. In order to keep the interval of such trajectories timelike (interval real) so that the local velocity of the fluid element does not exceed the light velocity, it is obvious that from Eq. (D-9),

$$0 \leq u^2 \leq 1 \ . \tag{D-10}$$

It is natural to identify (uc) with the local velocity of the fluid element at the point x^μ because $e^{M/2} d\sigma$ is the physical distance that the fluid element moves in a time $\frac{1}{c} dx^0$. Now define

$$\gamma = \frac{1}{(1 - u^2)^{1/2}} \tag{D-11}$$

so that

$$U^0 = \frac{dx^0}{ds} = \gamma \ . \tag{D-12}$$

Therefore,

$$U^j = \frac{dx^j}{ds} = \frac{dx^j}{dx^0} \frac{dx^0}{ds} = \frac{dx^j}{dx^0} \gamma \tag{D-13}$$

But Eq. (D-8) can be written as

$$u^2 = e^M \left[\left(\frac{dr}{dx^0}\right)^2 + r^2 \left(\frac{d\theta}{dx}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{dx^0}\right)^2 \right] . \tag{D-14}$$

If the quantities u_r, u_θ and u_ϕ are defined such that

$$u_r = e^{M/2} \frac{dr}{dx^0}$$

$$u_\theta = re^{M/2} \frac{d\theta}{dx^0}$$

$$u_\phi = r \sin\theta \; e^{M/2} \frac{d\phi}{dx^0} \qquad \text{(D-15)}$$

then

$$u^2 = u_r{}^2 + u_\theta{}^2 + u_\phi{}^2 \qquad \text{(D-16)}$$

It is natural to identify u_j as the j^{th} component of the local velocity at x^μ. Hence,

$$U^0 = \gamma$$

$$U^1 = \gamma e^{-M/2} u_r$$

$$U^2 = \gamma \frac{e^{-M/2}}{r} u_\theta$$

$$U^3 = \gamma \frac{e^{-M/2}}{r \sin\theta} u_\phi \; . \qquad \text{(D-17)}$$

By Eq. (D-6),

$$U_0 = \gamma$$

$$U_1 = -\gamma e^{M/2} u_r$$

$$U_2 = -\gamma r e^{M/2} u_\theta$$

$$U_3 = -\gamma r \sin\theta \; e^{M/2} u_\phi \; . \qquad \text{(D-18)}$$

Then Eq. (D-5) becomes,

$$T^0{}_0 = D + \frac{P}{c^2} u^2$$

$$T^1{}_1 = - Du_r^2 - \frac{P}{c^2} (1 + u_r^2 - u^2)$$

$$T^2{}_2 = - Du_\theta^2 - \frac{P}{c^2} (1 + u_\theta^2 - u^2)$$

$$T^3{}_3 = - Du_\phi^2 - \frac{P}{c^2} (1 + u_\phi^2 - u^2)$$

$$T^0{}_1 = - e^M T^1{}_0 = - (D + \frac{P}{c^2}) e^{M/2} u_r$$

$$T^0{}_2 = - r^2 e^M T^2{}_0 = - (D + \frac{P}{c^2}) r e^{M/2} u_\theta$$

$$T^0{}_3 = - r^2 \sin^2\theta \; e^M T^3{}_0 = - (D + \frac{P}{c^2}) r \sin\theta \; e^{M/2} u_\phi$$

$$T^1{}_2 = r^2 T^2{}_1 = - (D + \frac{P}{c^2}) r u_r u_\theta$$

$$T^1{}_3 = r^2 \sin^2\theta \; T^3{}_1 = - (D + \frac{P}{c^2}) r \sin\theta \; u_r u_\phi$$

$$T^2{}_3 = \sin^2\theta \; T^3{}_2 = - (D + \frac{P}{c^2}) \sin\theta \; u_\theta u_\phi \qquad \text{(D-19)}$$

where $D = \bar{D}\gamma^2$

$P = \bar{P}\gamma^2$.

To produce Eqs.(D-19), the following identities are needed.

$$\gamma^2 - 1 = \gamma^2 u^2$$

$$\gamma^2 u_j^2 + 1 = \gamma^2 (1 + u_j^2 - u^2) \quad . \qquad \text{(D-20)}$$

APPENDIX E

Solution of Cosmological Equations with a $1/R^m$ Source Term

The two usual cosmological equations of general relativity with zero cosmological constant but with a $1/R^m$ source term are [1]

$$\kappa\rho + \kappa\rho_V = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{E-1}$$

and

$$-\frac{\kappa P}{c^2} - \frac{\kappa P_V}{c^2} = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \tag{E-2}$$

where

$$\rho_V = \frac{V}{R^m} \tag{E-3}$$

$$P = \alpha c^2 \rho \tag{E-4}$$

$$P_V = \alpha_V c^2 \rho_V \tag{E-5}$$

$$\kappa = -\frac{8\pi G}{c^2} \tag{E-6}$$

$$' = \frac{d}{dx^0} \quad . \tag{E-7}$$

The difference between Eqs. (E-1) and (E-2) is

$$\kappa(\rho + \rho_V) + \frac{\kappa}{c^2}(P + P_V) + \frac{2k}{R^2} = 2\left(\frac{R'}{R}\right)' \quad . \tag{E-8}$$

The derivative of Eq. (E-1) is

$$\kappa(\rho' + \rho_V') = \frac{6kR'}{R^3} - 6\left(\frac{R'}{R}\right)\left(\frac{R'}{R}\right)' \quad . \tag{E-9}$$

The substitution of Eq. (E-8) into Eq. (E-9) gives

$$(\rho'+\rho_V') + \frac{3R'}{R}(\rho+\rho_V) + \frac{3R'}{R}\frac{(P+P_V)}{c^2} = 0 \quad . \tag{E-10}$$

If Eq. (E-8) is multiplied by R^3, it becomes

$$\left((\rho+\rho_V)R^3\right)' + \frac{(P+P_V)}{c^2}(R^3)' = 0 \quad . \tag{E-11}$$

This is the conservation of mass equation where ρ_V and P_V are the mass density and pressure of the source respectively. It is valid for any equation of state $P(\rho)$ and $P_V(\rho_V)$.

But with Eq. (E-5), $P_V = \alpha_V c^2 \rho_V$ and Eq. (E-4), $P = \alpha c^2 \rho$ Eq. (E-11) becomes

$$(\rho R^3)' + \alpha\rho(R^3)' = -(\rho_V R^3)' - \alpha_V \rho_V (R^3)' \quad . \tag{E-12}$$

The homogeneous equation associated with this equation is

$$(\rho R^3)' + \alpha\rho(R^3)' = 0, \tag{E-13}$$

the solution of which is

$$\rho = \frac{A}{R^{3(\alpha+1)}} \quad . \tag{E-14}$$

In order to solve Eq. (E-12), Eq. (E-14) may be substituted into it assuming A is now a function of x^0. Eq. (E-12) becomes

$$A' = -\rho_V' R^{3\alpha+3} - 3R^{3\alpha+2}R'\rho_V(1+\alpha_V) \quad . \tag{E-15}$$

With $\rho_V = \frac{V}{R^m}$, Eq. (E-15) becomes

$$A' = -\frac{\left(m-3(1+\alpha_v)\right)}{\left(m-3(1+\alpha)\right)}\, V\left(R^{3(\alpha+1)-m}\right)' \quad . \tag{E-16}$$

But Eq. (E-16) is easily integrable. It turns out that

$$A = -\, CVR^{3(\alpha+1)-m} + B \tag{E-17}$$

where

$$C = \frac{m-3(1+\alpha_v)}{m-3(1+\alpha)} \quad . \tag{E-18}$$

Finally, the substitution of Eq. (E-17) into (E-14) gives

$$\rho = \frac{B}{R^{3(\alpha+1)}} - \frac{CV}{R^m} \quad . \tag{E-19}$$

The substitution of Eq. (E-19) into Eq. (E-1) gives an expression for R'^2.

$$R'^2 = -\frac{\kappa}{3}\left[\frac{B}{R^{3\alpha+1}} + \frac{V(1-C)}{R^{m-2}}\right] - k \quad . \tag{E-20}$$

Thus,

$$\kappa\rho = -\frac{3a}{R^{3(\alpha+1)}} - \frac{f}{R^m} \tag{E-21}$$

and

$$R'^2 = \frac{a}{R^{3\alpha+1}} + \frac{b}{R^{m-2}} - k \tag{E-22}$$

where

$$a = -\frac{\kappa B}{3} \tag{E-23}$$

$$f = \kappa CV \tag{E-24}$$

$$b = \frac{\kappa V(C-1)}{3} \quad . \tag{E-25}$$

If $V = 0$ (no source term), then the solutions become the usual cosmological solutions for ρ and R. For a radiation

filled universe ($\alpha = 1/3$) [29] ,

$$\kappa\rho = - \frac{3a_r}{R^4} \tag{E-26}$$

and

$$R'^2 = \frac{a_r}{R^2} - k \, . \tag{E-27}$$

For a pressure free universe ($\alpha = 0$), the zero source term solutions are

$$\kappa\rho = - \frac{3a_d}{R^3} \tag{E-28}$$

and

$$R'^2 = \frac{a_d}{R} - k \, . \tag{E-29}$$

It should be noted that the meaning of the constant a is different in the case of the radiation model (denoted by a_r) than in the case of the pressure free model (denoted by a_d).

APPENDIX F

Elimination of the Spatial Derivatives from the Dynamic Equations for a Radiation Filled Universe

The terms involving the spatial partial derivatives of τ can be removed from Eqs. (9-1) and (9-2) by the substitution of Eq. (9-3). The result is

$$\kappa\rho - \frac{3}{2}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle + \frac{3k}{R^2}\langle\tau^2\rangle$$

$$= \frac{12k}{R^2} + \frac{15R''}{R} + \frac{12R'^2}{R^2} \tag{F-1}$$

$$-\kappa\rho - \frac{3}{2}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle - \frac{1}{2}\,\partial_0^2\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle + \frac{k}{R^2}\langle\tau^2\rangle$$

$$= \frac{4k}{R^2} + \frac{3R''}{R} + \frac{4R'^2}{R^2}\ . \tag{F-2}$$

But the right hand side of Eqs. (F-1) and (F-2) are not the expressions that would result from substitution of the average metric into the Einstein field tensor. In order to bring these equations into such a form, Eq. (F-2) must first be multiplied by minus five and added to Eq. (F-1). If this sum is multiplied by three eighths, then the desired form results.

$$\kappa\rho + \frac{15}{16}\,\partial_0^2\langle\tau^2\rangle + \frac{9}{4}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle - \frac{9}{8}\langle(\partial_0\tau)^2\rangle$$

$$-\frac{3}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{3k}{R^2} - \frac{3R'^2}{R^2}\ . \tag{F-3}$$

To obtain the other equation, Eq. (F-1) must be multiplied by minus 5/3 and Eq. (F-2) by 3. If these two equations are added and divided by eight, the resulting equation is formed.

$$- \frac{\kappa\rho}{3} - \frac{3}{16} \partial_0^2 \langle\tau^2\rangle - \frac{1}{4} \frac{R'}{R} \partial_0 \langle\tau^2\rangle + \frac{1}{8} \langle(\partial_0\tau)^2\rangle$$

$$- \frac{1}{4} \frac{k}{R^2} \langle\tau^2\rangle = - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \quad . \qquad \text{(F-4)}$$

The term $\partial_0^2 \langle\tau^2\rangle$ can be replaced by using Eq. (9-4). The final pair of equations are

$$\kappa\rho + \frac{5}{4} \kappa\langle\tau d\rangle + \frac{3}{4} \langle(\partial_0\tau)^2\rangle + \frac{3}{8} \frac{R'}{R} \partial_0 \langle\tau^2\rangle$$

$$- \frac{3}{4} \frac{k}{R^2} \langle\tau^2\rangle = - \frac{3k}{R^2} - \frac{3R'^2}{R^2} \qquad \text{(F-5)}$$

$$- \frac{\kappa\rho}{3} - \frac{1}{4} \kappa\langle\tau d\rangle - \frac{1}{4} \langle(\partial_0\tau)^2\rangle + \frac{1}{8} \frac{R'}{R} \partial_0 \langle\tau^2\rangle$$

$$- \frac{1}{4} \frac{k}{R^2} \langle\tau^2\rangle = - \frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \quad . \qquad \text{(F-6)}$$

APPENDIX G

Elimination of the Spatial Derivatives from the Dynamic Equations for a Dust Filled Universe

The elimination of the terms which contain the spatial partial derivatives in Eqs. (9-48), (9-49) and (9-50) gives

$$-\frac{3}{2}\kappa\rho - \frac{3}{2}\frac{R'}{R}\partial_0\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle - \frac{5}{2}\kappa\langle\tau d\rangle + \frac{3k}{R^2}\langle\tau^2\rangle = \frac{12k}{R^2} = \frac{12R'^2}{R^2} + \frac{15R''}{R} \tag{G-1}$$

$$-\frac{5}{6}\kappa\rho - \frac{3}{2}\frac{R'}{R}\partial_0\langle\tau^2\rangle - \frac{1}{2}\partial_0^2\langle\tau^2\rangle + \frac{3}{4}\langle(\partial_0\tau)^2\rangle - \frac{5}{6}\kappa\langle\tau d\rangle + \frac{k}{R^2}\langle\tau^2\rangle = \frac{4k}{R^2} + \frac{4R'^2}{R^2} + \frac{3R''}{R}\ . \tag{G-2}$$

Now, if Eq. (G-2) is multiplied by minus five and added to Eq. (G-1) and the resulting equation multiplied by three eighths, the following equation is obtained.

$$\kappa\rho + \frac{9}{4}\frac{R'}{R}\partial_0\langle\tau^2\rangle + \frac{15}{16}\partial_0^2\langle\tau^2\rangle - \frac{9}{8}\langle(\partial_0\tau)^2\rangle + \frac{5}{8}\kappa\langle\tau d\rangle - \frac{3}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{3k}{R^2} - \frac{3R'^2}{R^2}\ . \tag{G-3}$$

If Eq. (G-1) is multiplied by minus five thirds and Eq. (G-2) by three, and if these two resulting equations are added and divided by eight, the following equation is obtained.

$$- \frac{1}{4}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle - \frac{3}{16}\,\partial_0^2\langle\tau^2\rangle + \frac{1}{8}\langle(\partial_0\tau)^2\rangle + \frac{5}{24}\,\kappa\langle\tau d\rangle$$

$$- \frac{1}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \quad . \tag{G-4}$$

Now, the term $\partial_0^2\langle\tau^2\rangle$ that appears in Eqs. (G-3) and (G-4) can be eliminated by using Eq. (9-51). The result is

$$\kappa\rho + \frac{3}{4}\langle(\partial_0\tau)^2\rangle + \frac{3}{8}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle + \frac{5}{4}\,\kappa\langle\tau d\rangle$$

$$- \frac{3}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{3k}{R^2} - \frac{3R'^2}{R^2} \tag{G-5}$$

$$- \frac{1}{4}\langle(\partial_0\tau)^2\rangle + \frac{1}{8}\frac{R'}{R}\,\partial_0\langle\tau^2\rangle + \frac{1}{12}\,\kappa\langle\tau d\rangle$$

$$- \frac{1}{4}\frac{k}{R^2}\langle\tau^2\rangle = -\frac{k}{R^2} - \frac{2R''}{R} - \frac{R'^2}{R^2} \quad . \tag{G-6}$$

APPENDIX H

The Dynamics of the Energy Momentum Tensor for Disordered Radiation

The dynamics of the disordered radiation fluid is described by Eq. (9-10)

$$T^{\mu\nu}{}_{;\nu} = 0$$

where

$$T^{\mu\nu}{}_{;\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\nu}_{\nu\alpha} T^{\mu\alpha} + \Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} \quad . \tag{H-1}$$

The components of $T^{\mu}{}_{\nu}$ are given in Appendix D, and the components of $\Gamma^{\nu\mu}_{\alpha}$ are given by Eq. (5-10). Because of the summation which appears in Eq. (H-1) with respect to the indices ν, Eq. (H-1) represents a set of four differential equations. Adler [1] points out that the equation with μ equal to zero is the energy-conservation equation while the remaining three are the relativistic equivalent of Newton's law of dynamics applied to an ideal fluid. The conservation of energy is therefore expressed by

$$\partial_\nu T^{0\nu} + \Gamma^{\nu}_{\nu\alpha} T^{0\alpha} + \Gamma^{0}_{\alpha\nu} T^{\alpha\nu} = 0 \quad . \tag{H-2}$$

With the appropriate substitutions from Appendix D and Eq. (5-10), Eq. (H-2) becomes

$$\partial_0 (D + \frac{P}{c^2} u^2) + \frac{1}{2} (\partial_0 M)[(D + \frac{P}{c^2})(3 + u^2)]$$

$$+ e^{-M/2} \partial_r [(D + \frac{P}{c^2}) u_r] \qquad \text{(H-3) cont.}$$

$$+ e^{-M/2}(\partial_r M + \frac{2}{r})[(D + \frac{P}{c^2})u_r] \qquad \text{cont. (H-3)}$$

$$+ e^{-M/2} \frac{1}{r} \partial_\theta[(D + \frac{P}{c^2})u_\theta]$$

$$+ e^{-M/2}(\frac{1}{r} \partial_\theta M + \frac{\cot\theta}{r})[(D + \frac{P}{c^2}) u_\theta]$$

$$+ e^{-M/2} \frac{1}{r\sin\theta} \partial_\phi[(D + \frac{P}{c^2}) u_\phi]$$

$$+ e^{-M/2} (\frac{1}{r\sin\theta} \partial_\phi M)[(D + \frac{P}{c^2}) u_\phi] = 0 .$$

If the equation of state given by Eq. (5-22) is used, Eq. (H-3) becomes,

$$\partial_0 D + 2\partial_0 MD + \frac{1}{3} \partial_0(Du^2) + \frac{2}{3} (\partial_0 M)Du^2$$

$$+ \frac{4}{3} [e^{-M/2}\partial_r(Du_r) + e^{-M/2}(\partial_r M + \frac{2}{r})(Du_r)$$

$$+ e^{-M/2} \frac{1}{r} \partial_\theta(Du_\theta) + e^{-M/2} (\frac{1}{r} \partial_\theta M + \frac{\cot\theta}{r})(Du_\theta)$$

$$+ e^{-M/2} \frac{1}{r\sin\theta} \partial_\phi(Du_\phi)$$

$$+ e^{-M/2} (\frac{1}{r\sin\theta} \partial_\phi M)(Du_\phi)] = 0. \qquad \text{(H-4)}$$

In order to find suitable expressions that might replace the correlation functions $<\tau d>$ and $<\tau\partial_0 d>$ in Eqs. (9-4) and (9-5), the following procedure is used.

If Eq. (H-4) is multiplied by a special function $h(\tau,\partial_0\tau)$ and averaged it becomes

$$<h\partial_0 D> + 2<h(\partial_0 M)D> + \frac{1}{3} <h\partial_0(Du^2)>$$

$$+ \frac{2}{3} \langle hDu^2(\partial_0 M)\rangle + 4\langle e^{-M/2} \frac{h}{r \sin\theta} \partial_\phi (Du_\phi)\rangle$$

$$+ 4\langle e^{-M/2} \frac{hDu_\phi}{r \sin\theta} (\partial_\phi M)\rangle = 0 \ . \qquad \text{(H-5)}$$

It should be noted that in Eq. (H-5) only partial derivatives with respect to x^0 and ϕ appear explicitly. Those terms which contain partial derivatives with respect to r and θ are buried in the terms containing the ϕ partial derivatives. To see how this can be done, consider the following argument. In Eq. (H-4), there are three types of terms involving spatial partial derivative with respect to r, θ, and ϕ. On a spatial hypersurface of constant x^0 it is reasonable to assume that the average of a quantity associated with a given direction in space ought to be equal to the average of that same quantity associated with a different direction. Since the r, θ, and ϕ terms of Eq. (H-4) have identical roles, (i.e. transport of mass in r, θ, ϕ directions as Adler [1] points out) then the average of these terms (multiplied by the function h which has no explicit spatial dependence) should be equal.

Now, D can be decomposed into its average and fluctuating components. That is,

$$D = \bar{\rho}(x^0) + \bar{d}(x^0, r, \theta, \phi; \xi) \qquad \text{(H-6)}$$

such that $\langle D\rangle = \bar{\rho}$.

If Eq. (H-5) is expressed in terms of τ instead of M, it becomes, using Eq. (H-6)

$$\partial_0\bar{\rho}\langle h\rangle + \langle h\partial_0\bar{d}\rangle + \frac{4R'}{R}\bar{\rho}\langle h\rangle + \frac{4R'}{R}\langle h\bar{d}\rangle$$

$$+ 2\bar{\rho}\langle h\left(\frac{\partial_0\tau}{1+\tau}\right)\rangle + 2\langle h\left(\frac{\partial_0\tau}{1+\tau}\right)\bar{d}\rangle + \frac{1}{3}\partial_0\bar{\rho}\langle hu^2\rangle$$

$$+ \frac{1}{3}\bar{\rho}\langle h\partial_0(u^2)\rangle + \frac{1}{3}\langle hu^2(\partial_0\bar{d})\rangle + \frac{1}{3}\langle h\bar{d}(\partial_0 u^2)\rangle$$

$$+ \frac{4}{3}\bar{\rho}\frac{R'}{R}\langle hu^2\rangle + \frac{2}{3}\bar{\rho}\langle hu^2\left(\frac{\partial_0\tau}{1+\tau}\right)\rangle$$

$$+ \frac{4}{3}\frac{R'}{R}\langle h\bar{d}u^2\rangle + \frac{2}{3}\langle h\bar{d}u^2\left(\frac{\partial_0\tau}{1+\tau}\right)\rangle$$

$$+ 4\langle e^{-M/2}\frac{u_\phi h}{r\sin\theta}\partial_\phi\bar{\rho}\rangle + 4\bar{\rho}\langle e^{-M/2}\frac{h}{r\sin\theta}\partial_\phi u_\phi\rangle$$

$$+ 4\langle e^{-M/2}\frac{hu_\phi}{r\sin\theta}\partial_\phi\bar{d}\rangle + 4\langle e^{-M/2}\frac{h\bar{d}}{r\sin\theta}\partial_\phi u_\phi\rangle$$

$$+ 4\bar{\rho}\langle e^{-M/2}\frac{hu_\phi}{r\sin\theta}\left(\frac{\partial_\phi\tau}{1+\tau}\right)\rangle$$

$$+ 4\langle e^{-M/2}\frac{h\bar{d}u_\phi}{r\sin\theta}\left(\frac{\partial_\phi\tau}{1+\tau}\right)\rangle = 0 \,. \qquad \text{(H-7)}$$

It should be expected that weak fluctuations of geometry do not create pressure gradients which accelerate the radiation fluid to local velocities near the speed of light. That is,

$$u^2 \ll 1$$

whenever

$$\tau^2 \ll 1 \,. \qquad \text{(H-8)}$$

This makes certain moments of Eq. (H-7) small compared to others.

Consider now two cases. Let h symbolically stand for two special functions, τ and $\partial_0\tau$. Then

$$\langle h \rangle = 0 \; . \tag{H-9}$$

Thus, if the higher order moments of Eq. (H-7) are neglected by virtue of the smallness of u^2, this equation becomes

$$\langle h\partial_0 \bar{d} \rangle + \frac{4R'}{R} \langle h\bar{d} \rangle + 2\bar{\rho}\langle h\partial_0 \tau \rangle + \frac{1}{3}\bar{\rho}\langle h\partial_0 u^2 \rangle$$

$$+ \frac{2}{3}\bar{\rho}\langle hu^2\partial_0 \tau \rangle + 4\bar{\rho}\langle e^{-M/2} \frac{h}{r \sin\theta} \partial_\phi u_\phi \rangle$$

$$+ 4\langle e^{-M/2} \frac{hu_\phi}{r \sin\theta} \partial_\phi \bar{d} \rangle + 4\bar{\rho}\langle e^{-M/2} \frac{hu_\phi}{r \sin\theta} \partial_\phi \tau \rangle$$

$$= 0 \; . \tag{H-10}$$

As pointed out by Harrison and others [9], any perturbation of the radiation fluid characterized by d, u_r, u_θ, u_ϕ will propagate with the sound velocity of the fluid. For radiation this velocity is near the speed of light.

$$\left(V_{\text{sound}} = \sqrt{\frac{dP}{dD}} = \frac{1}{\sqrt{3}} \;\; c \right)$$

Hence, such an accoustical signal would propagate along "near" geodesic lines. That is, ds^2 (accoustical signal) is near enough to zero that

$$dx^0 \simeq e^{M/2} d\sigma \simeq e^{M/2} \; r \sin\theta \; d\phi \; \cdot \tag{H-11}$$

But this implies that the three moments which contain the spatial partial derivatives with respect to ϕ in Eq. (H-10) ought to be of the same order as those for which $e^{-M/2} \times \frac{1}{r \sin\theta} \partial\phi$ is replaced by ∂_0. For example,

$$\langle e^{-M/2} \frac{h}{r \sin\theta} \partial_\phi u_\phi \rangle \simeq \langle h\partial_0 u_\phi \rangle \quad .$$

Then, Eq. (H-10) becomes after discarding the insignificant moments

$$\langle h\partial_0\bar{d}\rangle + \frac{4R'}{R}\langle h\bar{d}\rangle + 2\bar{\rho}\langle h\partial_0\tau\rangle$$

$$+ 4\bar{\rho}\langle e^{-M/2}\frac{h}{r\sin\theta}\partial_\phi u_\phi\rangle = 0 \ . \qquad \text{(H-12)}$$

If the first two terms of Eq. (H-12) are consolidated into one term $f_1\langle h\partial_0\bar{d}\rangle$ (for an appropriate f_1), it becomes

$$f_1\langle h\partial_0\bar{d}\rangle + 4\bar{\rho}\langle e^{-M/2}\frac{h}{r\sin\theta}\partial_\phi u_\phi\rangle$$

$$= -2\bar{\rho}\langle h\partial_0\tau\rangle \ . \qquad \text{(H-13)}$$

If the characteristic time for the fluctuation is comparable to the characteristic time $\frac{1}{c}\frac{R}{R'}$ of the average geometry, then the second term is equal in magnitude to the first. In this case, f_1, which is a function of x^0, is of order one.

Now consider the remaining equations $T^{j\nu};\nu = 0$ for j equal to one, two, and three. After some rather tedious calculations, these equations become

$(T^{1\nu};\ \nu = 0)$

$$\partial_0[(D + \frac{P}{c^2})\, e^{-M/2}u_r] + (\frac{5}{2}\,\partial_0 M)[(D + \frac{P}{c^2})\, e^{-M/2}u_r]$$

$$+ \partial_r\{e^{-M}[Du_r^2 + \frac{P}{c^2}(1 + u_r^2 - u^2)]\}$$

$$+ (\frac{5}{2}\,\partial_r M + \frac{3}{r})\, e^{-M}[Du_r^2 + \frac{P}{c^2}\,(1 + u_r^2 - u^2)]$$

$$- \frac{1}{2}\,(\partial_r M + \frac{2}{r})\, e^{-M}[Du^2 + \frac{P}{c^2}\,(3 - 2u^2)] \qquad \text{(H-14) cont.}$$

$$+ \partial_\theta[(D + \frac{P}{c^2}) \frac{e^{-M}}{r} u_r u_\theta] \qquad \text{cont.}$$

$$+ (\frac{5}{2} \partial_\theta M + \cot\theta)[(D + \frac{P}{c^2}) \frac{e^{-M}}{r} u_r u_\theta]$$

$$+ \partial_\phi[(D + \frac{P}{c^2}) \frac{e^{-M}}{r \sin\theta} u_r u_\phi]$$

$$+ \frac{5}{2} (\partial_\phi M)[(D + \frac{P}{c^2}) \frac{e^{-M}}{r \sin\theta} u_r u_\phi] = 0 \qquad \text{(H-14)}$$

$(T^{2\nu};\ \nu = 0)$

$$\partial_0[(D + \frac{P}{c^2}) \frac{e^{-M/2}}{r} u_\theta] + (\frac{5}{2} \partial_0 M)[(D + \frac{P}{c^2}) \frac{e^{-M/2}}{r} u_\theta]$$

$$+ \partial_r [(D + \frac{P}{c^2}) \frac{e^{-M}}{r} u_r u_\theta]$$

$$+ (\frac{5}{2} \partial_r M + \frac{4}{r})[(D + \frac{P}{c^2}) \frac{e^{-M}}{r} u_r u_\theta]$$

$$+ \frac{1}{r} \partial_\theta \left\{ \frac{e^{-M}}{r} [Du_\theta^2 + \frac{P}{c^2} (1 + u_\theta^2 - u^2)] \right\}$$

$$+ (\frac{5}{2} \frac{1}{r} \partial_\theta M + \frac{\cot\theta}{r}) \left\{ \frac{e^{-M}}{r} [Du_\theta^2 + \frac{P}{c^2} (1 + u_\theta^2 - u^2)] \right\}$$

$$- \frac{1}{2} (\frac{1}{r} \partial_\theta M) \left\{ \frac{e^{-M}}{r} [Du^2 + \frac{P}{c^2} (3 - 2u^2)] \right.$$

$$- \frac{\cot\theta}{r} \left\{ \frac{e^{-M}}{r} [Du_\phi^2 + \frac{P}{c^2} (1 - u_\phi^2 - u^2)] \right\}$$

$$+ \frac{1}{r \sin\theta} \partial_\phi[(D + \frac{P}{c^2}) \frac{e^{-M}}{r} u_\theta u_\phi] \qquad \text{(H-15) cont.}$$

$$+ \left(\frac{5}{2} \frac{1}{r \sin\theta} \partial_\phi M\right)\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M}}{r} u_\theta u_\phi\right] = 0 \qquad \text{cont. (H-15)}$$

$(T^{3\nu};\ \nu = 0)$

$$\partial_0\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M/2}}{r \sin\theta} u_\phi\right] + \left(\frac{5}{2} \partial_0 M\right)\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M/2}}{r \sin\theta} u_\phi\right]$$

$$+ \partial_r\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M}}{r \sin\theta} u_r u_\phi\right]$$

$$+ \left(\frac{5}{2} \partial_r M + \frac{4}{r}\right)\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M}}{r \sin\theta} u_r u_\theta\right]$$

$$+ \frac{1}{r} \partial_\theta\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M}}{r \sin\theta} u_\theta u_\phi\right]$$

$$+ \left(\frac{5}{2} \frac{1}{r} \partial_\theta M + \frac{3 \cot\theta}{r}\right)\left[\left(D + \frac{P}{c^2}\right) \frac{e^{-M}}{r \sin\theta} u_\theta u_\phi\right]$$

$$+ \frac{1}{r \sin\theta} \partial_\phi \left\{ \frac{e^{-M}}{r \sin\theta} \left[D u_\phi^2 + \frac{P}{c^2} (1 + u_\phi^2 - u^2)\right] \right\}$$

$$+ \left(\frac{5}{2} \frac{1}{r \sin\theta} \partial_\phi M\right) \left\{ \frac{e^{-M}}{r \sin\theta} \left[D u_\phi^2 + \frac{P}{c^2} (1 - u_\phi^2 - u^2)\right] \right\}$$

$$- \left(\frac{1}{2} \frac{1}{r \sin\theta} \partial_\phi M\right) \left\{ \frac{e^{-M}}{r \sin\theta} \left[D u^2 + \frac{P}{c^2} (3 - 2u^2)\right] \right\}$$

$$= 0 \,. \qquad \text{(H-16)}$$

If the u^2 terms are discarded and if the equation of state is used, these equations become simply

$$\partial_0 (D u_r) + \frac{4R'}{R} D u_r + \frac{2\partial_0 \tau}{1+\tau} (D u_r) = -\frac{1}{4} e^{-M/2} \partial_r D$$

$$\text{(H-17) cont.}$$

$$\partial_0 (D u_\theta) + \frac{4R'}{R} D u_\theta + \frac{2\partial_0 \tau}{1+\tau} (D u_\theta) = -\frac{1}{4} e^{-M/2} \frac{1}{r} \partial_\theta D$$

$$\partial_0(Du_\phi) + \frac{4R'}{R} Du_\phi + \frac{2\partial_0\tau}{1+\tau}(Du_\phi)$$

$$= -\frac{1}{4} e^{-M/2} \frac{1}{r \sin\theta} \partial_\phi D \ . \qquad \text{(H-17)}$$

In order to learn something about the behavior of the moment $\langle e^{-M/2} \frac{h}{r \sin\theta} \partial_\phi u_\phi \rangle$ which appears in Eq. (H-13), consider the average of the third of Eq. (H-17) multiplied by h.

$$\bar{\rho}\langle h\partial_0 u_\phi\rangle = -\frac{1}{4}\langle e^{-M/2} \frac{h}{r \sin\theta} \partial_\phi \bar{d}\rangle \qquad \text{(H-18)}$$

(To obtain Eq. (H-18), the higher order moments are neglected.).

Because of the previous arguments concerning the relationship of spatial and temporal partial derivatives, the following equation should hold for some f_2 of order one.

$$\bar{\rho}\langle e^{-M/2} \frac{h}{r \sin\theta} \partial_\phi u_\phi\rangle = f_2\langle h\partial_0\bar{d}\rangle \ . \qquad \text{(H-19)}$$

If Eq. (H-19) is substituted into Eq. (H-13), the following equation results.

$$\langle h\partial_0\bar{d}\rangle = \left(\frac{-2}{f_1 + 4f_2}\right) \bar{\rho}\langle h\partial_0\tau\rangle \ . \qquad \text{(H-20)}$$

If the changes of $\bar{d}$ and τ occur over the same time, it must be true that

$$\langle h\bar{d}\rangle = f_3\bar{\rho}\langle h\tau\rangle \qquad \text{(H-21)}$$

where f_3 is a weak function of x^0 of order one (unless by chance $f_1 + 4f_2$ is small).

But in Eqs. (9-4) and (9-5), the correlations $<\tau d>$ and $<(\partial_0\tau)d>$ are required. Since the ratio of D to $\bar{D}$ is $(1 + u^2/3\ \ )$ (see Eq. (9-6)) which is very close to one, it can be assumed that for some functions λ_f and β_f, both weak functions of x^0 and of order one as described above, the following equations hold.

$$<\tau d> = -\ 2\lambda_f\rho<\tau^2>$$

$$<(\partial_0\tau)d> = -\ \beta_f\rho\partial_0<\tau^2>. \qquad \text{(H-22)}$$

If λ_f and β_f are expanded in a Taylor series and only the constant terms are retained because of the supposed weak dependence of these functions on x^0, the following approximate equations result.

$$<\tau d> = -\ 2\lambda\rho<\tau^2>$$

$$<(\partial_0\tau)d> = -\ \beta\rho\partial_0<\tau^2>. \qquad \text{(H-23)}$$

These constants are then to be viewed as arbitrary parameters of the model. Eqs. (H-23) are therefore consistent with the conservation equations.

APPENDIX I

The Dynamics of the Energy Momentum Tensor for a Matter Filled Universe

The dynamics of the energy-momentum distribution is given by

$$T^{\mu\nu}{}_{;\,\nu} = 0. \tag{I-1}$$

The conservation of energy equation corresponding to μ equal to zero is given by Eq. (H-3) with u_j and P equal to zero. (It should be noted that u_j and P equal to zero with D unspecified are solutions to the three momentum equations for which $\mu = 1, 2, 3$. See Eqs. (H-14), (H-15), and (H-16).) That is,

$$\partial_0 D + \frac{3}{2}(\partial_0 M)D = 0 \quad . \tag{I-2}$$

If this equation is multiplied by $e^{3/2M}$ and integrated, the following equation results.

$$D = \frac{A}{e^{3M/2}} \quad . \tag{I-3}$$

With

$$D = \rho + d \tag{I-4}$$

and

$$e^{3/2M} = e^{3/2G}\,(1+\tau)^{3/2} \quad , \tag{I-5}$$

equation (I-3) becomes for small τ

$$\rho + d = \frac{A}{e^{3/2G}}\,(1 - \frac{3}{2}\tau). \tag{I-6}$$

Hence,

$$\rho = \frac{A}{e^{3G/2}} \tag{I-7}$$

and

$$d = -\frac{3}{2}\rho\tau \quad . \tag{I-8}$$

That is, for a pressure free universe, the stochastic process d is a function of the stochastic prosess τ. Hence,

$$<\tau d> = -\frac{3}{2}\rho<\tau^2> \tag{I-9}$$

and

$$<(\partial_0\tau)d> = -\frac{3}{4}\rho\partial_0<\tau^2> \quad . \tag{I-10}$$

APPENDIX J

Solution of the Growth Equations for a Radiation Filled Universe

If $y = \langle\tau^2\rangle$, $z = \bar{R}^4 \dfrac{\langle(\partial_0\tau)^2\rangle}{a_r}$, $x = R/R^+$, the growth Eqs. (9-34) and (9-35) are

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 8\lambda y = 2x^4 z \tag{J-1}$$

and

$$\frac{dz}{dx} + \frac{4}{x} z = \frac{4\beta}{x^4} \frac{dy}{dx} \tag{J-2}$$

where R^+ is a constant with the dimension of length. All the variables (y, z, x) in Eqs. (J-1) and (J-2) are dimensionless.

The solution of these two coupled differential equations proceeds as follows. If Eq. (J-2) is multiplied by x^4, the left hand side becomes a derivative of the function $x^4 z$. This equation may then be integrated directly to give

$$y = \frac{1}{4\beta} x^4 z + f_g \tag{J-3}$$

where f_g is an arbitrary constant of integration.

From Eq. (J-3), the first and second derivatives of y with respect to x may be found in terms of z and x. If these are substituted into Eq. (J-1), a second order differential equation in z results.

$$x^2 \frac{d^2z}{dx^2} + 9x \frac{dz}{dx} + 8(2 - \lambda - \beta)z = \frac{32\beta\lambda f_g}{x^4} . \qquad (J-4)$$

Equation (J-4) can be transformed into a second order differential equation with constant coefficients by the change of variable

$$x = e^t . \qquad (J-5)$$

Thus,

$$\frac{d^2z}{dt^2} + 8 \frac{dz}{dt} + 8(2 - \lambda - \beta)z = 32\beta\lambda f_g e^{-4t} . \qquad (J-6)$$

The associated homogeneous equation of Eq. (j-6) is

$$\frac{d^2z}{dt^2} + 8 \frac{dz}{dt} + 8(2 - \lambda - \beta)z = 0 .$$

Its complementary solution z_c is

$$z_c = Ae^{it} + Be^{jt} \qquad (J-8)$$

where $i = -4 + 2\sqrt{2(\lambda+\beta)}$

$j = -4 - 2\sqrt{2(\lambda+\beta)}$

and λ and β are arbitrary constants.

The complementary solution z_c can be used to find the general solution to Eq. (J-6). This is done by assuming the general solution is given by Eq. (J-8) but with A and B functions of t.

If this solution form is substituted into Eq. (J-6), a differential equation in A and B results.

$$A \left[\frac{d^2}{dt^2} + 8 \frac{d}{dt} + 8(2 - \lambda - \beta) \right] e^{it} \qquad (J-9)$$

cont.

$$+ B\left[\frac{d^2}{dt^2} + 8\frac{d}{dt} + 8(2-\lambda-\beta)\right] e^{jt} \qquad \text{(J-9)}$$

$$+ \left[\frac{dA}{dt}\frac{d(e^{it})}{dt} + \frac{dB}{dt}\frac{d(e^{jt})}{dt}\right]$$

$$+ \frac{d}{dt}\left[(e^{it})\frac{dA}{dt} + e^{jt}\frac{dB}{dt}\right] + 8\left[(e^{it})\frac{dA}{dt} + e^{jt}\frac{dB}{dt}\right]$$

$$= 32\beta\lambda f_g e^{-4t} . \qquad \text{(J-9)}$$

The first two terms are zero by virtue of the fact that e^{it} and e^{jt} are solutions to Eq. (J-7). In the equation,

$$z(t) = A(t)e^{it} + B(t)e^{jt} , \qquad \text{(J-10)}$$

three dependent variables of t are present: z, A, and B. But in general, three variables can be made to satisfy three equations. However, so far, only two such equations have been specified, namely Eq. (J-4) and Eq. (J-10). Hence, a third equation can be freely chosen. A convenient choice is

$$e^{it}\frac{dA}{dt} + e^{jt}\frac{dB}{dt} = 0 \qquad \text{(J-11)}$$

so that Eq. (J-9) becomes

$$\frac{d(e^{it})}{dt}\frac{dA}{dt} + \frac{d(e^{jt})}{dt}\frac{dB}{dt} = 32\beta\lambda f_g e^{-4t} . \qquad \text{(J-12)}$$

These two equations can be written in matrix form.

$$\tilde{Z}\tilde{u} = \tilde{F} \qquad \text{(J-13)}$$

where $\tilde{Z} = \begin{pmatrix} e^{it} & e^{jt} \\ \frac{d(e^{it})}{dt} & \frac{d(e^{jt})}{dt} \end{pmatrix}$

$$\tilde{u} = \begin{pmatrix} \frac{dA}{dt} \\ \\ \frac{dB}{dt} \end{pmatrix}$$

$$\tilde{F} = 32\beta\lambda f_g e^{-4t} \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \quad .$$

By Cramer's Rule,

$$\frac{dA}{dt} = \frac{1}{\det \tilde{Z}} \det \begin{bmatrix} 0 & e^{jt} \\ \\ 32\beta\lambda f_g e^{-4t} & \frac{d(e^{jt})}{dt} \end{bmatrix}$$

$$\frac{dB}{dt} = \frac{1}{\det \tilde{Z}} \det \begin{bmatrix} e^{it} & 0 \\ \\ \frac{d(e^{it})}{dt} & 32\lambda\beta f_g e^{-4t} \end{bmatrix} \tag{J-14}$$

so that

$$\frac{dA}{dt} = -\frac{32\lambda\beta f_g e^{-(i+4)t}}{j-i} \tag{J-15}$$

$$\frac{dB}{dt} = \frac{32\lambda\beta f_g e^{-(j+4)t}}{j-i} \tag{J-16}$$

for $\quad j \neq i \; (\lambda = -\beta)$.

Now, Eqs. (J-15) and (J-16) may be integrated to give

$$A = \frac{32\lambda\beta f_g}{(j-i)(i+4)} e^{-(i+4)t} + \overline{D} \tag{J-17}$$

$$B = -\frac{32\lambda\beta f_g}{(j-i)(j+4)} e^{-(j+4)t} + \overline{E} \tag{J-18}$$

where $\overline{D}$ and $\overline{E}$ are arbitrary constants. Hence, by Eq. (J-10) and by the identity $x = e^t$,

$$z = -4\lambda F x^{-4} + \bar{D}x^{i} + \bar{E}x^{j} \tag{J-19}$$

where $F = \frac{\beta f_g}{\lambda+\beta}$ and is a new arbitrary constant. By Eq. (J-3),

$$y = F + \frac{\bar{D}}{4\beta}x^{i+4} + \frac{\bar{E}}{4\beta}x^{j+4} \ . \tag{J-20}$$

If these solutions for y and z are substituted into Eqs. (J-1) and (J-2), they are found to satisfy these equations. If y and z of Eqs. (J-19) and (J-20) are translated back into $\langle\tau^2\rangle$ and $\langle(\partial_0\tau)^2\rangle$, Eqs. (J-19) and (J-20) become

$$\langle\tau^2\rangle = F_{1/3} + \frac{I_{1/3}}{4\beta}R^{i+4} + \frac{D_{1/3}}{4\beta}R^{j+4} \tag{J-21}$$

$$\frac{\langle(\partial_0\tau)^2\rangle}{a_r} = -4\lambda F_{1/3}R^{-4} + I_{1/3}R^{i} + D_{1/3}R^{j} \tag{J-22}$$

where $i = -4 + 2\sqrt{2(\lambda+\beta)}$

$j = -4 - 2\sqrt{2(\lambda+\beta)}$

and $I_{1/3}$, $D_{1/3}$, and $F_{1/3}$ are arbitrary constants. ($I_{1/3} = \bar{D}/R^{+i+4}$ and $D_{1/3} = \bar{E}/R^{+j+4}$.)

APPENDIX K

Solution of the Growth Equations for a Matter Filled Universe

If $y = \langle\tau^2\rangle$; $z = \frac{R^{+3}}{a_d} \langle(\partial_0\tau)^2\rangle$; $x = \frac{R}{R^{+}}$ (for some R^+), the growth Eqs. (9-65) and (9-66) are

$$x^2y'' + \frac{3}{2}xy' - 3y = 2x^3z \tag{K-1}$$

$$z' + \frac{4}{x}z = \frac{3}{2}\frac{1}{x^3}y' \quad . \tag{K-2}$$

If Eq. (K-1) is differentiated and Eq. (K-2) is substituted into it, a third order differential equation for z results.

$$2x^3z''' + 27x^2z'' + 84xz' + 42z = 0 \ . \tag{K-3}$$

If a change of variables is made such that

$$x = e^t \tag{K-4}$$

then Eq. (K-3) becomes

$$2z_{ttt} + 21z_{tt} + 61z_t + 42z = 0 \ . \tag{K-5}$$

The general solution of this equation is simply,

$$z = \bar{G}e^{-t} + \bar{H}e^{-7t/2} + \bar{J}e^{-6t} \quad . \tag{K-6}$$

In terms of x, Eq. (K-6) becomes

$$z = \bar{G}x^{-1} + \bar{H}x^{-7/2} + \bar{J}x^{-6} \ . \tag{K-7}$$

If this equation is substituted into Eq. (K-2), an equation for y' results. If this equation is integrated, the

following expression for y appears.

$$y = \bar{G}x^2 - \frac{2\bar{H}}{3} x^{-1/2} + \frac{4}{9} \bar{J}x^{-3} + \bar{K} . \tag{K-8}$$

But Eqs. (K-7) and (K-8) are general solutions to the set of the two coupled Eqs. (K-3) and (K-2). Four integrations are required to solve this system which yields four arbitrary constants ($\bar{G}$, $\bar{H}$, $\bar{J}$, and $\bar{K}$ above). However, the given system of equations (Eq. (K-1) and (K-2)) require only three integrations. Because Eq. (K-3) is derived from these equations by differentiation, Eqs. (K-7) and (K-8) must be the general solution to the given equations with one of the appropriate arbitrary constants equal to zero. By inspection, the equation

$$y = \bar{K} \text{ and } z = 0$$

although a solution to Eqs. (K-3) and (K-2), is not a solution to the original equations. Hence, the general solution to Eqs. (K-1) and (K-2) must be

$$z = \bar{G}x^{-1} + \bar{H}x^{-7/2} + \bar{J}x^{-6} \tag{K-9}$$

$$y = \bar{G}x^2 - \frac{2\bar{H}}{3} x^{-1/2} + \frac{4}{9} \bar{J}x^{-3} . \tag{K-10}$$

Hence,

$$\frac{\langle(\partial_0\tau)^2\rangle}{a_d} = I_0R^{-1} + F_0R^{-7/2} + D_0R^{-6} \tag{K-11}$$

and

$$\langle\tau^2\rangle = I_0R^2 - \frac{2F_0}{3} R^{-1/2} + \frac{4}{9} D_0R^{-3} . \tag{K-12}$$

APPENDIX L

Classification of Solutions for the Radiation Filled Universe

With the exception of the integration of R'^2, Eqs. (E-20) and (E-21) of Appendix E give the solutions to Eqs. (E-1) and (E-2). These solutions are, with k neglected as usual (see Eqs. (9-41) to (9-45)),

$$\kappa\rho = -\frac{3a_r}{R^4} - f_A R^{m_A} \tag{L-1}$$

$$R'^2 = a_r R^{-2} + b_A R^{m_A+2} \tag{L-2}$$

where

$$f_{I_{1/3}} = \frac{C(I_{1/3})}{12}\left[10\lambda + 4\beta - 1 + \frac{2\beta}{\sqrt{2(\lambda+\beta)}}\right] \tag{L-3}$$

$$f_{D_{1/3}} = \frac{C(D_{1/3})}{12}\left[10\lambda + 4\beta - 1 - \frac{2\beta}{\sqrt{2(\lambda+\beta)}}\right] \tag{L-4}$$

$$b_{I_{1/3}} = -\frac{C(I_{1/3})}{36}\left[\frac{2\lambda + \sqrt{2(\lambda+\beta)}}{\sqrt{2(\lambda+\beta)}}\right] \tag{L-5}$$

$$b_{D_{1/3}} = \frac{C(D_{1/3})}{36}\left[\frac{2\lambda - \sqrt{2(\lambda+\beta)}}{\sqrt{2(\lambda+\beta)}}\right] \tag{L-6}$$

$$m_{I_{1/3}} = i = -4 + 2\sqrt{2(\lambda+\beta)} \tag{L-7}$$

$$m_{D_{1/3}} = j = -4 - 2\sqrt{2(\lambda+\beta)}\quad . \tag{L-8}$$

The A subscript designates the type of fluctuation. ($I_{1/3}$ - type corresponds to a situation where the nonuniformity

increases with increasing R and $D_{1/3}$ - type corresponds to a situation where the nonuniformity decreases with inceasing R.)

There are four kinds of solutions based upon the manner in which $\kappa\rho$ and R are displaced from the fluctuation free solutions $\kappa\rho^*$ and R^*. That is, f_A and b_A can in principle be each positive or negative depending on the choice of λ and β.

In order to determine how a given (λ,β) displace $\kappa\rho$ and R from $\kappa\rho^*$ and R^*, it is necessary to locate those values (λ,β) for which f and b are equal to zero. Then a set (λ,β) which lies on a particular side of the two curves of zero f_A and zero b_A determines the sign of f_A and b_A and hence the direction of displacement of $\kappa\rho$ and R from $\kappa\rho^*$ and R^*.

In order to find the line of zero f_A in λ-β space, consider first the following definition.

$$n = \sqrt{2(\lambda+\beta)} \quad . \tag{L-9}$$

The set (λ,β) for which f_A equals zero is given by the equation

$$10\lambda + 4\beta - 1 = \pm \frac{2\beta}{\sqrt{2(\lambda+\beta)}} \tag{L-10}$$

where the plus sign is used if A represents $D_{1/3}$ and the minus sign if A represents $I_{1/3}$. This equation as it stands is rather difficult to plot directly. But the plotting is simplified if Eq. (L-10) is separated into two parametric equations with parameter n.

The two parametric equations $\beta(n)$ and $\lambda(n)$ are found to be

$$\beta = \frac{n}{2}\left[\frac{5n^2 - 1}{3n \pm 1}\right] \tag{L-11}$$

$$\lambda = -\frac{n}{2}\left[\frac{2n^2 \mp n - 1}{3n \pm 1}\right] \tag{L-12}$$

where $n \geq 0$. The top sign corresponds to the $D_{1/3}$ - type fluctuation and the bottom sign to the $I_{1/3}$ - type fluctuation.

By varying n, the equation $\beta(\lambda)$ can be traced out. This equation is plotted in Figs. 3 and 4 for both $I_{1/3}$ and $D_{1/3}$ type fluctuations respectively.

The equation of λ vs. β for $b_A = 0$ is much simpler to find. With straightforward algebra, this equation is

$$\beta = \lambda(2\lambda - 1) \tag{L-13}$$

for both the $I_{1/3}$ and $D_{1/3}$ - type fluctuations. This equation is also plotted in Figs. 3 and 4. It should be noted that Eq. (L-13) is quadratic in λ. Hence, according to this equation, there should be two values of λ corresponding to a given value of β. However, only one value satisfies the requirement that b_A equal zero. That there are two values of λ comes from the fact that Eq. (L-13) is derived by a squaring operation which thereby introduces an extra root.

APPENDIX M

Index and Location of Symbols Used in Dissertation

The following list contains the symbols used in this work. Accompanying each symbol is the location where the definition or explanation of the symbol may be found.

1.	L^*, G, c	I
2.	ds, $g_{\mu\nu}$, x^μ, x^0, r, θ, d, $d\sigma^2$, R_G,	IV-A
	R_M, k, e^G, e^M, M	IV-A
3.	$f_{\mu\nu}$, f_M, τ	IV-B
4.	$[\mu\nu,\beta]$, ∂_μ, $\Gamma^\alpha_{\mu\nu}$, $g^{\alpha\beta}$, $R_{\mu\nu}$, $R^\lambda{}_\nu$, R,	
	$G^\lambda{}_\nu$, κ	V-A
5.	$\overline{D}$, $\overline{P}$, u, D, P, α	V-F
6.	R	VI-A
7.	ρ, d	IX-A, B,
8.	M, V, ΔD, ΔM, ΔV, g, $\overline{V}$, λ, β	IX-A-2
9.	$F_{1/3}$, $I_{1/3}$, $D_{1/3}$, i, j	IX-A-3
10.	$C(I_{1/3})$, $C(D_{1/3})$, $C(A)$	IX-A-4-a
11.	ρ^*, R^*	IX-A-4-b
12.	F_0, I_0, D_0	IX-B-3
13.	$B(I_0)$, $B(D_0)$	IX-B-4
14.	$\delta\rho$, D, I, T_I, T_B, $\langle\tau^2\rangle_i$, $\langle\tau^2\rangle_0$, $\overline{R}$	X-A

15. δT, T, H_R, L, $\delta\rho_0$, ρ_0, ε, R_0, R_{max}, R_e X-B

16. ν, $\vec{V}$, $\vec{v}$, P, p A

17. $F(\delta;x^\mu)$, $f(D;x^\mu)$, $h[D(x^\mu)]$ B

18. $T^{\mu\nu}$, $T^\lambda{}_\nu$, U^μ, U_ν, γ D

19. ρ_V, P_V, α_V, V, m, κ, $(')$, C, B, a, a_r,
a_d, f, b E

20. $\bar{\rho}$, $\bar{d}$, h, f_1, f_2, f_3, λ_f, β_f H

21. y, x, z, R^+, t J, K

22. t, A, B, $\tilde{Z}$, $\tilde{u}$, $\tilde{F}$, f_g, $\bar{D}$, $\bar{E}$, F J

23. $\bar{G}$, $\bar{H}$, $\bar{J}$, $\bar{K}$ K

24. m_A, $f_{I_{1/3}}$, $f_{D_{1/3}}$, $b_{I_{1/3}}$, $b_{D_{1/3}}$, $m_{I_{1/3}}$,
$m_{D_{1/3}}$, f_A, b_A, n L

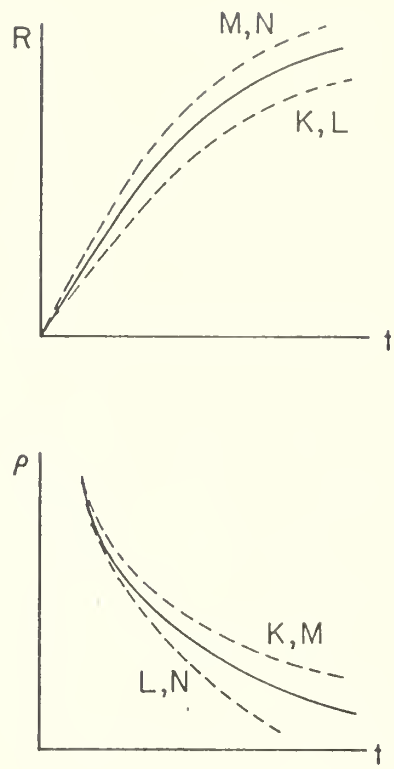

Figure 1. Classification Designations
for the Increasing Type Fluctuations.

The solid lines are the fluctuation free solutions for R and ρ. The broken lines depict the possible deviations of R and ρ from these unperturbed solutions. The letters K, L, M, and N label the four possible combinations of perturbed solutions for R and ρ from their unperturbed values.

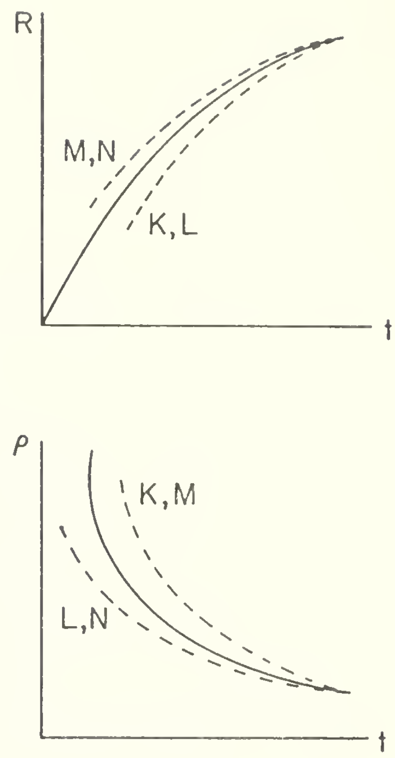

Figure 2. Classification Designations for the Decreasing Type Fluctuations.

The solid lines are the fluctuation free solutions for R and ρ. The broken lines depict the possible deviations of R and ρ from these unperturbed solutions. The letters K, L, M, and N label the four possible combinations of perturbed solutions for R and ρ from their unperturbed values.

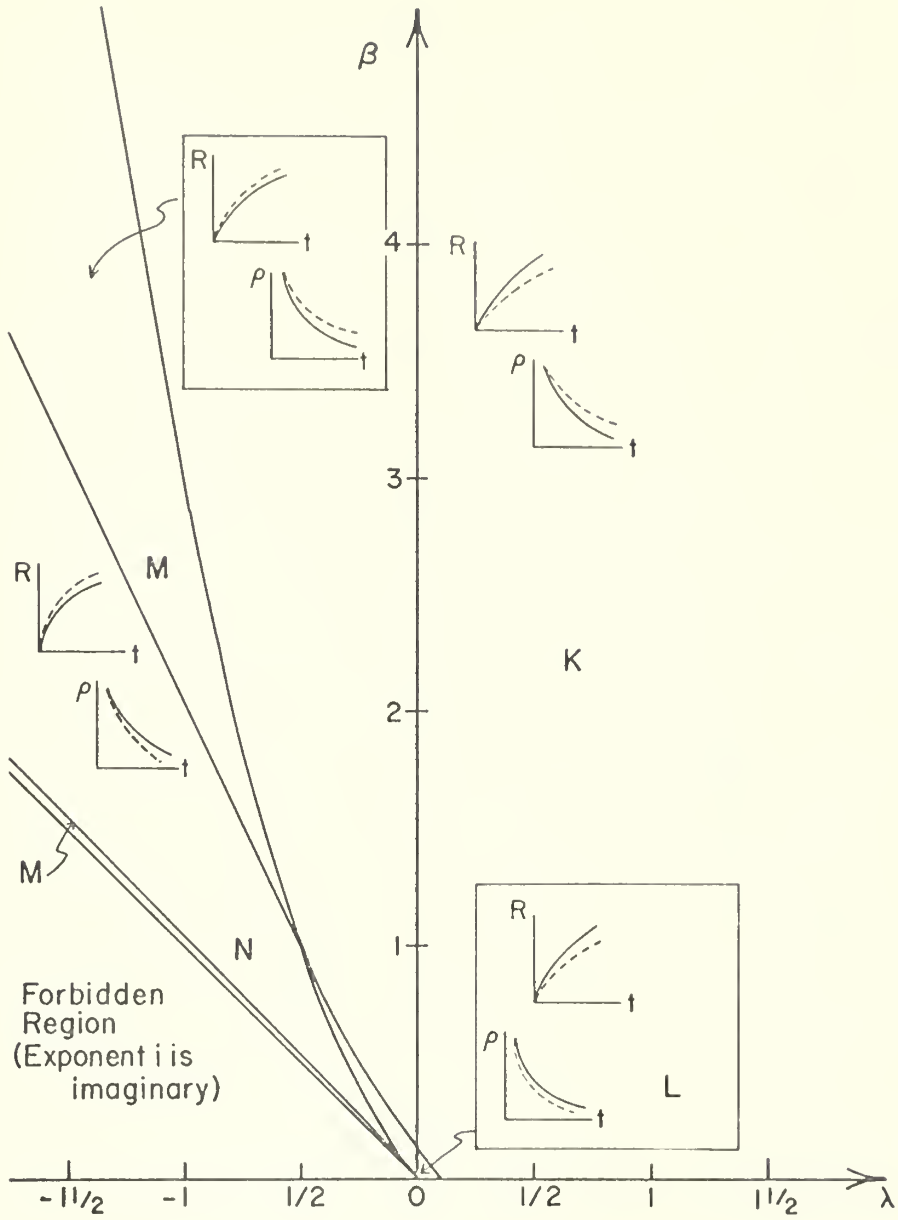

Figure 3. Solution Types in λ-β Space for the Increasing Type Fluctuations for the Radiation Filled Universe

The λ-β plane is sectioned into four regions. The perturbed solutions for each region for R and ρ are depicted as broken lines. The fluctuation free solutions are represented as solid lines.

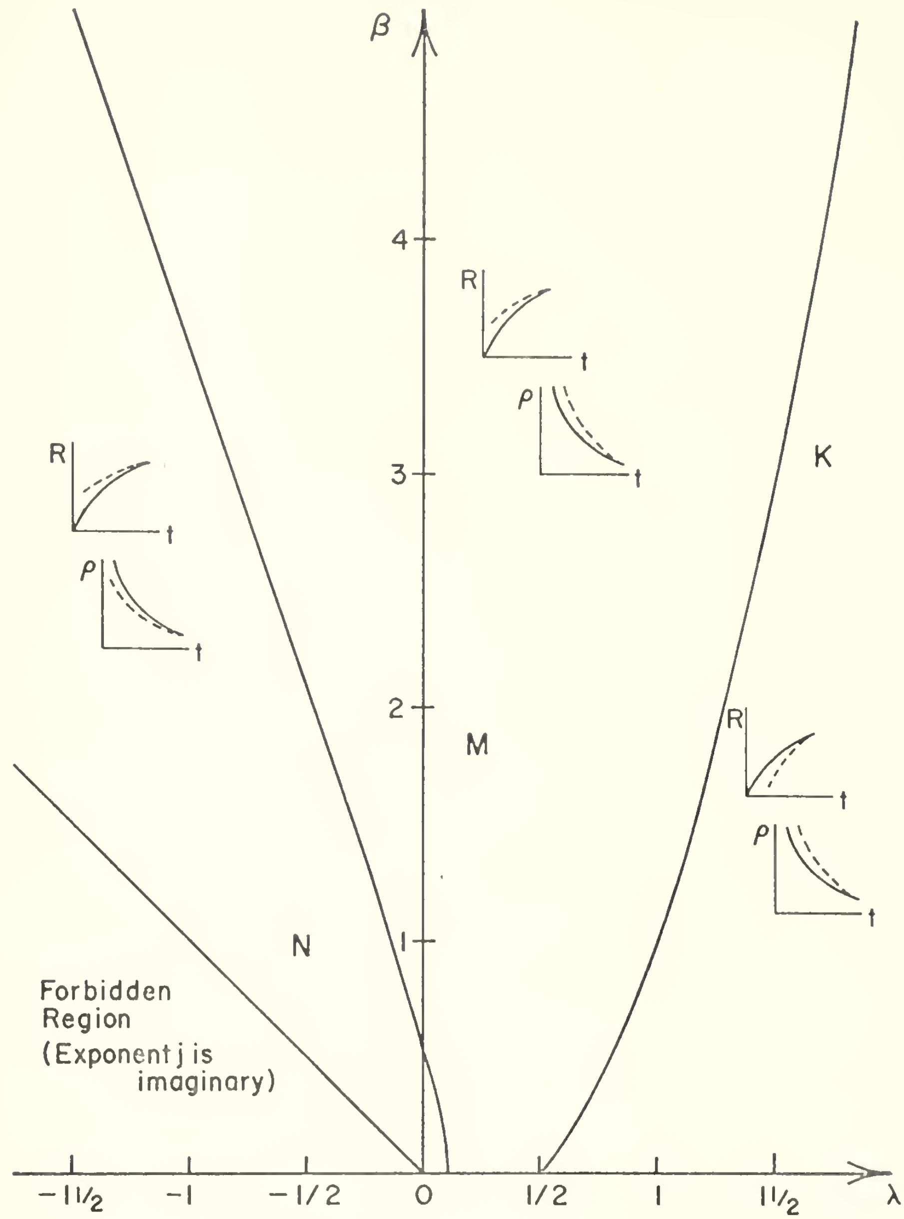

Figure 4. Solution Types in λ-β Space for the Decreasing Type Fluctuations for the Radiation Filled Universe. The λ-β plane is sectioned into three regions. The perturbed solutions for each region for R and ρ are depicted as broken lines. The fluctuation free solutions are represented as solid lines.

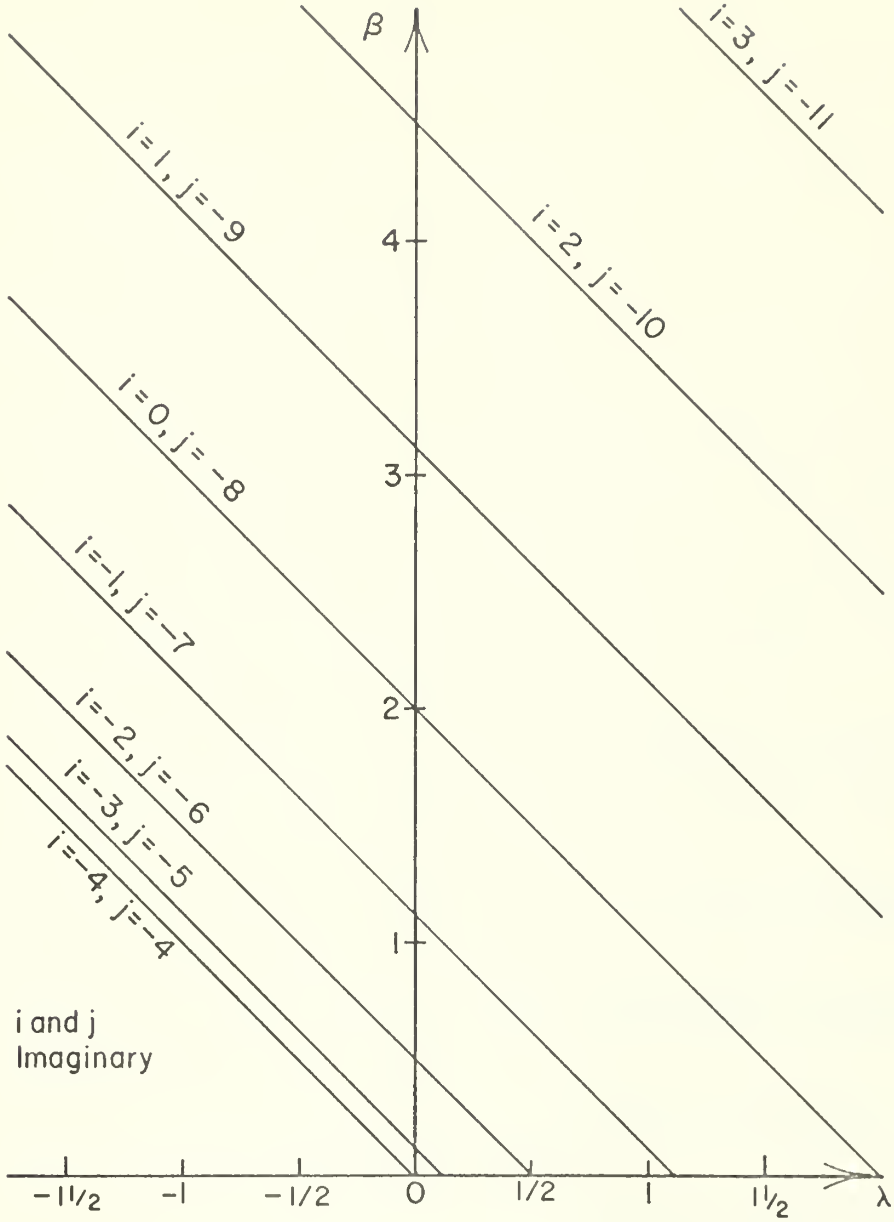

Figure 5. Lines of Constant i and j.

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13. ABSTRACT

Uniform and isotropic mathematical models of the expanding universe usually predict an initial singularity of infinite mass density and space curvature. To study possible mechanisms which would avoid the occurance of these singularities, non-uniform cosmological models based on Einstein's field equations are investigated in which random perturbations of long wave lengths are superimposed on the Robinson-Walker metric of the unperturbed models. Techniques of fluid turbulence theory, used to describe random fields by a hierarchy of central moments of the random perturbations, are applied to describe the dynamics of these moments. For the case of small perturbations the hierarchy is truncated and solutions are found. The solutions are either growing or decaying perturbations leading to R^m extra terms in the usual cosmological equations for the curvature radius R. The result agrees with the small perturbation Fourier series expansion analysis which exists in the literature. Based on the upper limit of the anisotropy of the 3° K background radiation, the growing perturbation model predicts a maximum expansion even for k=0, Euclidean spaces. The decaying perturbation solutions give extra terms of the form $1/R^m$ with $m>4$ in the cosmological equations and indicate that the mechanism of long wave random perturbations may prevent the original singularity and make oscillatory models possible.

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